# THE THEORY OF
# GROUP CHARACTERS
## AND MATRIX REPRESENTATIONS OF GROUPS

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# PREFACE

Since the discovery of group characters by Frobenius at the end of the last century, the development of the theory has been so spectacular, and the theory has shown such powerful contacts with other branches of mathematics, both pure and applied, that the inadequacy of its treatment by text-books is rather surprising. Indeed, until the publication last year of Murnaghan's treatise, *The Theory of Group Representations*, there was no book which devoted itself especially to the theory, and even Murnaghan's work was written specifically with a view to its applications to quantum theory and nuclear physics.

It has been my purpose in writing this book to give a simple and self-contained exposition of the theory in relation to both finite and continuous groups, and to develop some of its contacts with other branches of pure mathematics, such as invariant theory, group theory, and the theory of symmetric functions. There are three introductory chapters on matrices, algebras, and groups, so that no specialized knowledge is required of the reader beyond that obtained in an ordinary degree course in mathematics. Rather than to attempt any exhaustive treatment, it has been my aim to develop the nucleus of a theory which will bring to notice new problems to be solved.

The bibliography gives most of the original memoirs which have gone towards the development of the theory, together with text-books and authoritative references to relevant theories. I must express my debt to Murnaghan's book (no. 9 in the bibliography) and to two books by Weyl (nos. 13 and 14) in compiling this bibliography. Murnaghan's book also suggested certain additions to the last chapter.

I have to thank Prof. A. R. Richardson for his suggestions and comments concerning the writing of this book, Dr. A. J. Ward and Dr. A. C. Aitken for invaluable help in reading the proofs, and Mr. H. O. Foulkes for some corrections to the tables of characters.

D. E. L.

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## CORRIGENDUM

*p*. 23. **The regular matrix representation**

This representation will not be *simply* isomorphic if there exists an element $x$ of the algebra for which $ax = 0$ for all $a$ of the algebra. The corresponding matrix $X$ would be identically zero. A simply isomorphic representation, however, may be obtained in any case by adjoining a modulus to the algebra before obtaining the regular representation.

# MATRICES

## 1.1. Linear transformations

CONSIDER the set of $n$ linear equations

$$\begin{aligned} x_1' &= a_{11}x_1+a_{12}x_2+...+a_{1n}x_n, \\ x_2' &= a_{21}x_1+a_{22}x_2+...+a_{2n}x_n, \\ &\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot \\ x_n' &= a_{n1}x_1+a_{n2}x_2+...+a_{nn}x_n, \end{aligned}$$

which may be written more concisely

$$x_p' = \sum a_{pq}x_q. \tag{1.1;1}$$

The coefficients $a_{pq}$, for the purposes of this book, are taken to be any complex numbers. More generally, they may be taken from any prescribed field, but the theory is simplified in the case of the complex numbers, owing to the fact that the $n$th degree equation in one variable has exactly $n$ roots in this field; and this case is sufficient for the purposes of this book. For the more general theory the reader is referred to Turnbull and Aitken, *The Theory of Canonical Matrices* (London and Glasgow, 1932).

The equations (1.1; 1) are said to form a *linear transformation* in $n$ variables. They may be regarded as effecting a mapping of an $n$-dimensional space into itself, i.e., relative to any assigned Cartesian coordinate system, the point $P' \equiv (x_1', x_2',..., x_n')$ is made to correspond to the point $P \equiv (x_1, x_2,..., x_n)$. The point $P'$ is uniquely defined by $P$. This $n$-dimensional space is called the *carrier space.*

If the determinant of the coefficients is not zero, the equations (1.1; 1) may be solved for the $x_r$'s in the form†

$$x_q = \sum a_{qp}' x_p' \tag{1.1;2}$$

and the point $P$ is also uniquely defined by the point $P'$. Such a transformation is called *non-singular*; it effects a *bi-uniform mapping.* The transformation (1.1; 2) is called the *inverse transformation* of (1.1; 1).

---

† *Vide* any standard text-book on algebra, e.g. Chrystal.

## 1.2. Matrices

The transformation is completely defined by the $n^2$ quantities $a_{pq}$. We therefore associate with the transformation the array of numbers

$$\begin{bmatrix} a_{11}, & a_{12}, & a_{1n} \\ a_{21}, & & a_{2n} \\ \\ a_{n1}, & a_{n2}, & a_{n} \end{bmatrix}$$

This array is called a *matrix*, the *matrix of the transformation.* We write it shortly $[a_{st}]$, where $s$ is the index of the row, and $t$ the index of the column from which the given element is chosen. The letters $s$ and $t$ will in general be reserved throughout this book to indicate the row and the column of a matrix from which a typical element is chosen.

Two matrices are said to be equal if and only if they are identical, i.e.

$$[a_{st}] = [b_{st}]$$

if and only if $a_{pq} = b_{pq}$ for all $p$ and $q$.

Now let $x''_p \quad \sum b_{pq} x'_q$ be a second transformation with matrix of coefficients $[b_{st}]$. The effect of taking the two transformations consecutively in order of definition is clearly to produce a third transformation,

$$x''_p = \sum c_{pq} x_q,$$

where

$$c_{pq} = \sum b_{pr} a_{rq}.$$

The matrix of this transformation is $[c_{st}]$.

We therefore define the *product* of two matrices by the rule

$$[b_{st}][a_{st}] = [\sum b_{sr} a_{rt}] = [c_{st}]. \tag{1.2;1}$$

Thus if two transformations are taken consecutively, the matrix of the combined transformation is the product of the matrices.

From the usual rule for the product of two determinants† we see that:

*The determinant of the product of two matrices is equal to the product of the determinants.*

It should be noted that the product $[a_{st}][b_{st}] = [\sum a_{sr} b_{rt}]$ is not in general equal to $[b_{st}][a_{st}]$. Multiplication is *not commutative.* Multiplication is, however, associative, and it is easily verified that

$$\{[a_{st}][b_{st}]\}[c_{st}] = [a_{st}]\{[b_{st}][c_{st}]\}.$$

The matrix with unity in each position in the leading diagonal, and zero elsewhere, i.e. the matrix $[\delta_{st}]$, where $\delta_{ij} = 0$ $(i \neq j)$ and $\delta_{ii} = 1$, which corresponds to the identical transformation, is called the unit matrix and will be denoted by $I$. If it is desired to convey the order, $n^2$, of the matrices, it will be written $I_n$. Clearly

---

† See Chrystal.

$$[a_{st}]I = I[a_{st}] = [a_{st}].$$

If the determinant of $[a_{st}]$, which we shall denote by $|a_{st}|$, is not equal to zero, then, as we have stated, there exists an inverse transformation (1.1; 2), and hence a matrix $[a'_{st}]$, such that

$$[a_{st}][a'_{st}] = [a'_{st}][a_{st}] = I.$$

$[a'_{st}]$ is called the reciprocal of the matrix $[a_{st}]$, and is denoted by $[a_{st}]^{-1}$. In this case the matrix $[a_{st}]$ is said to be *non-singular*.

If, however, $|a_{st}| = 0$, then there is no reciprocal of $[a_{st}]$ and the matrix is said to be *singular*.

In addition to multiplication we define *addition* for matrices by the rule

$$[a_{st}]+[b_{st}] = [a_{st}+b_{st}].$$

Clearly addition is commutative and associative, and multiplication is distributive with respect to addition.

**Permutation matrix**

A matrix in which each row and each column has but one non-zero element, which is equal to unity, is called a *permutation matrix*. Transformation (see §1.3) by a permutation matrix has the effect of permuting the order of the rows of a matrix, and the columns also, in the same manner.

**Diagonal matrices**

If in a matrix $A = [a_{st}]$ we have

$$a_{ij} = 0 \quad (i \neq j),$$

the matrix is called a diagonal matrix. It is completely defined by the elements in the leading diagonal, namely $a_{11}, a_{22},..., a_{nn}$, and we shall use the concise notation

$$A = \mathrm{diag}(a_{11}, a_{22},..., a_{nn}).$$

The notation may be made even more concise, in the case where a given element is repeated in consecutive positions, by the use of indices to indicate repetitions. Thus

$$\mathrm{diag}([1]^3, [2]^2, [3]) = \begin{bmatrix} 1, & & & & & \\ & 1, & & & 0 & \\ & & 1, & & & \\ & & & 2, & & \\ & 0 & & & 2, & \\ & & & & & 3 \end{bmatrix}$$

The notation $\operatorname{diag}(A_1, A_2,..., A_r)$ will also be used when the $A_i$ represent square matrices, this signifying that the matrices $A_i$ are placed in symmetric positions about the leading diagonal.

## 1.3. The transform of a matrix

Consider the effect of a change of the coordinate system in the carrier space on a transformation

$$x'_p = \sum a_{pq} x_q.$$

Let $y_1, y_2,..., y_n$ be the coordinates of a point relative to the new system, with

$$x_p = \sum c_{pq} y_q,$$

the matrix $[c_{st}]$ being non-singular.

If $y'_1,..., y'_n$ are the new coordinates of the transformed point, then

$$x'_p = \sum c_{pq} y'_q.$$

Now let the original point transformation referred to the new coordinate system take the form

$$y'_p = \sum b_{pq} y_q.$$

The coefficients $b_{pq}$ may be found as follows:

$$x'_p = \sum a_{pq} x_q = \sum_{q,r} a_{pq} c_{qr} y_r,$$

$$x'_p = \sum c_{pq} y'_q = \sum_{q,r} c_{pq} b_{qr} y_r.$$

Thus

$$\sum_q a_{pq} c_{qr} = \sum_q c_{pq} b_{qr},$$

$$[a_{st}][c_{st}] = [c_{st}][b_{st}],$$

i.e.

$$[b_{st}] = [c_{st}]^{-1}[a_{st}][c_{st}]. \tag{1.3; 1}$$

The matrix $[b_{st}]$ is called the *transform of the matrix* $[a_{st}]$ *by the matrix* $[c_{st}]$.

Clearly $[a_{st}]$ is the transform of $[b_{st}]$ by the matrix $[c_{st}]^{-1}$. Matrices which are transforms of one another are called *equivalent* matrices. Since they may be regarded as corresponding to the same point transformation, but referred to different coordinate systems, equivalent matrices have many properties in common. In fact a very powerful method of finding the properties of a matrix is to find an equivalent matrix of simpler form, e.g. diagonal, and to find the properties of this second matrix. We shall pursue this method in the section on the 'classical canonical form'.

## 1.4. Rectangular matrices and vectors

In a set of linear equations

$$x'_p = \sum a_{pq} x_q,$$

the number $m$, of new variables $x'_p$, may differ from the number $n$, of original variables $x_q$. Correspondingly we have a *rectangular matrix* $[a_{st}]$ with $m$ rows and $n$ columns. The product of two rectangular matrices $[a_{st}]$, $[b_{st}]$ may be obtained according to the same rule (1.2; 1) as for square matrices, but it is necessary that the number of rows in $[a_{st}]$ should be equal to the number of columns in $[b_{st}]$. It may be convenient to make up the number of rows or columns by adding zeros.

Of particular importance are matrices with one row, or one column. These are called *vectors*. The coordinates of a point in $n$-space may be formed into a vector $[x_s]$, and the transformation (1.1; 1) may then be expressed by matrix multiplication

$$[x'_s] = [a_{st}][x_s],$$

a form which lends itself to great neatness of expression.

A set of $r$ vectors $X_1 = [x_{1s}]$, $X_2 = [x_{2s}]$,..., $X_r = [x_{rs}]$ are said to be *linearly dependent* if scalars, i.e. ordinary complex numbers, $\alpha_1, \alpha_2,..., \alpha_r$ can be found such that

$$\alpha_1 X_1 + \alpha_2 X_2 + ... + \alpha_r X_r = 0.$$

Otherwise they are *linearly independent*. It is clearly impossible to find a set of more than $n$ linearly independent vectors of order $n$.

A square matrix $[a_{st}]$ of order $n^2$ may be regarded as composed of $n$ column vectors

$$[a_{st}] = [A_t],$$

where

$$A_p = [a_{sp}].$$

If $|a_{st}| = 0$, the vectors $A_p$ are linearly dependent, since a non-zero solution of

$$\begin{aligned} \sum a_{1p}\,\alpha_p &= 0, \\ \sum a_{2p}\,\alpha_p &= 0, \\ \cdot \quad \cdot \quad &\cdot \quad \cdot \\ \sum a_{np}\,\alpha_p &= 0 \end{aligned}$$

may be found, treated as equations in $\alpha_1, \alpha_2,..., \alpha_n$. This condition may be expressed: there is a vector $[\alpha_t]$ such that $[\alpha_t][a_{st}] = 0$.

If, further, the $n$ vectors $[a_{sp}]$ are linearly dependent upon $j$ of the vectors, then there will be $n-j$ linearly independent vectors $[\alpha_{qt}]$ $(1 \leqslant q \leqslant n-j)$ such that

$$[\alpha_{qt}][a_{st}] = 0,$$

and the matrix $[a_{st}]$ is said to be of *rank* $j$.

The condition for this is clearly that all $(j+1)$-rowed minors of the matrix $[a_{st}]$ have zero determinant, and hence is the same as the condition that the row-vectors $[a_{qt}]$ should be linearly dependent upon $j$ of these. In this case we have also that $(n-j)$ linearly independent row-vectors $[\beta_{sp}]$ can be found such that

$$[a_{st}][\beta_{sp}] = 0 \quad (1 \leqslant p \leqslant n-j).$$

There is no difficulty in proving that the rank of a transform of a matrix is the same as the rank of the matrix. More generally, if $X$ and $Y$ are non-singular matrices, $A$ and $XAY$ have the same rank, for if $[\beta_{st}]$ is a rectangular matrix with $(n-j)$ columns and

$$Y^{-1}[\beta_{st}] = [\beta'_{st}],$$

then if

$$A[\beta_{st}] = 0,$$

also

$$XAY[\beta'_{st}] = 0.$$

### 1.5. The characteristic equation of a matrix

Let $A = [a_{st}]$ be any square matrix of order $n^2$, and $I$ the unit matrix of the same order. If $\lambda$ is a variable scalar, the matrix $[\lambda I-A]$ is singular for certain fixed values of $\lambda$ called the *characteristic roots* of the matrix $A$. These values may be found by equating to zero the determinant

$$|\lambda I-A| = 0.$$

We obtain the *characteristic equation*

$$\lambda^n-a_1\lambda^{n-1}+a_2\lambda^{n-2}-\ldots+(-1)^n a_n = 0,$$

$a_r$ being the sum of the determinants of the $r$-rowed principal (coaxial) minors, i.e. the minors in which the indices of the rows are the same as those of the columns, and $a_n$ being the determinant of the matrix $A$ itself.

Hence there are exactly $n$ characteristic roots of a matrix of order $n^2$, some of which may be repeated.

The characteristic equation of a matrix has the important property that it is invariant for transformations of the matrix.

*Equivalent matrices have the same characteristic equations.*

The proof is quite simple, for since

$$[\lambda I - T^{-1}ST] = [T^{-1}][\lambda I - S][T],$$

we have also

$$|\lambda I - T^{-1}ST| = |T^{-1}||\lambda I - S||T|$$
$$= |\lambda I - S|.$$

## 1.6. The classical canonical form of a matrix

### (a) All characteristic roots distinct

Of all the transforms of a given matrix, certain ones may be chosen as especially simple in form. These are the *canonical forms.* Of these the most important, and the only one we shall consider here, is the *classical canonical form,* which we now proceed to obtain. We treat first of the simpler case when all the characteristic roots of the matrix are distinct.

Let $A = [a_{st}]$ be a matrix of order $n^2$ with $n$ distinct characteristic roots $\lambda_1, \lambda_2,..., \lambda_n$.

Since $[\lambda_r I - A]$ is a singular matrix, we can find a vector $[b_{sr}]$ such that

$$[\lambda_r I - A][b_{sr}] = 0.$$

Hence

$$A[b_{sr}] = [b_{sr}]\lambda_r. \qquad (1.6;1)$$

Further, the $n$ vectors $[b_{sr}]$ are linearly independent, for since

$$(A - \lambda_r I)[b_{sr}] = 0,$$

if

$$\sum \alpha_r[b_{sr}] = 0,$$

multiplying on the left by $(A - \lambda_2 I)(A - \lambda_3 I)...(A - \lambda_n I)$ we should obtain

$$\alpha_1[b_{s1}](\lambda_1 - \lambda_2)(\lambda_1 - \lambda_3)...(\lambda_1 - \lambda_n) = 0.$$

Hence $\alpha_1 = 0$, and similarly $\alpha_2 = \alpha_3 = ... = \alpha_n = 0$.

Thus, since the vectors $[b_{sr}]$ are linearly independent, the matrix $[b_{st}]$ is non-singular. Combining all the equations (1.6; 1) in matrix form we obtain

$$A[b_{st}] = [b_{st}][\lambda_s \delta_{st}],$$

where $\delta_{ij} = 0$ $(i \neq j)$, and $\delta_{ii} = 1$. $[\lambda_s \delta_{st}]$ is a diagonal matrix.

But since $[b_{st}]$ is non-singular we may put

$$[b_{st}]^{-1}A[b_{st}] = [\lambda_s \delta_{st}]$$
$$= \mathrm{diag}(\lambda_1, \lambda_2,..., \lambda_n).$$

I. *Any matrix of which all the characteristic roots are distinct may be transformed into a diagonal matrix in which the diagonal elements are the characteristic roots of the matrix.*

This is the classical canonical form of the matrix for this case. It is clearly unique, save for the order in which the characteristic roots are placed in the leading diagonal.

From the existence of this form we deduce an important theorem, namely

*Every matrix satisfies its own characteristic equation.*

Firstly, the product of two diagonal matrices is obtain by multiplying corresponding terms, e.g.

$$\operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)\operatorname{diag}(\mu_1,\mu_2,\ldots,\mu_n) = \operatorname{diag}(\lambda_1\mu_1,\lambda_2\mu_2,\ldots,\lambda_n\mu_n).$$

It follows that if $A = \operatorname{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n)$,

then $$[A-\lambda_1 I][A-\lambda_2 I]\ldots[A-\lambda_n I] = 0,$$

i.e. $$A^n-a_1A^{n-1}+a_2A^{n-2}-\ldots+(-1)^n a_n I = 0,$$

and a diagonal matrix must satisfy its characteristic equation.

Secondly, since equivalent matrices have the same characteristic equation, and since also they must satisfy the same equation, for

$$(T^{-1}AT)^n = T^{-1}A^nT,$$

it follows that every matrix which can be transformed into a diagonal form, e.g. every matrix with distinct characteristic roots, must satisfy its characteristic equation.

Lastly, if $A$ is any matrix whatsoever, we can find a matrix $Z$ such that $A+\mu Z$ has all its characteristic roots distinct and satisfies its characteristic equation. We now take the limit as $\mu$ tends to zero, whence $A$ satisfies its characteristic equation.

## 1.7. The classical canonical form of a matrix

### (b) Multiple characteristic roots

Let $A = [a_{st}]$ be a matrix of order $n^2$ for which the characteristic root $\lambda_1$ is repeated $r_1$ times. Then a vector $[\alpha_{s1}]$ can be found such that

$$[a_{st}][\alpha_{s1}] = [\alpha_{s1}]\lambda_1.$$

If $[\alpha_{st}]$ is any non-singular matrix of which the first column is $[\alpha_{s1}]$, then the first column of $[a_{st}][\alpha_{st}]$ is $[\alpha_{s1}]\lambda_1$. Hence the first column of $[\alpha_{st}]^{-1}[a_{st}][\alpha_{st}]$ consists of $\lambda_1$ followed by $(n-1)$ zeros.

Let $$B = [\alpha_{st}]^{-1}[a_{st}][\alpha_{st}] = \left[\begin{array}{c|c} \lambda_1 & b_t \\ \hline 0 & b_{st} \end{array}\right],$$

in which $s$ and $t$ run from 2 to $n$.

Now $B$ has the characteristic root $\lambda_1$ repeated $r_1$ times, since it is a transform of $A$, and hence $[b_{st}]$ has the same root repeated $(r_1-1)$ times. Hence a matrix $[\beta_{st}]$ $(2 \leqslant s, t \leqslant n)$ can be found such that $[\beta_{st}]^{-1}[b_{st}][\beta_{st}]$ is of the form

$$\left[\begin{array}{c|c} \lambda_1 & c_t \\ \hline 0 & c_{st} \end{array}\right].$$

Thus the matrix $\left[\begin{array}{c|c} 1 & 0 \\ \hline 0 & \beta_{st} \end{array}\right]$ transforms $B$ into

$$\begin{matrix} \lambda_1 & & \alpha_t \\ & \lambda_1 & \end{matrix}$$

Proceeding thus, a matrix $K$ can be found such that

$$K^{-1}AK = \begin{matrix} \lambda_1, & \gamma_{12}, & \gamma_{13}, & \dots, & \gamma_{1n} \\ & \lambda_1, & \gamma_{23}, & \dots, & \gamma_{2n} \\ & & \ddots\lambda_1, & & \gamma_{r_1 n} \end{matrix}$$

Let $K_1$ be the rectangular matrix which consists of the first $r_1$ columns of $K$. Then

$$AK_1 = K_1 \begin{bmatrix} \lambda_1, & \gamma_{12}, & \gamma_{13}, & \dots, & \gamma_{1r_1} \\ & \lambda_1, & \gamma_{23}, & \dots, & \gamma_{2r_1} \\ & & & & \\ 0 & & \ddots\lambda_1, & \gamma_{r_1-1,r_1} & \\ & & & \lambda_1 & \end{bmatrix} = K_1 A_1.$$

Clearly $A_1-\lambda_1 I_{r_1}$ is a matrix with zeros on and below the leading diagonal, and
$$[A_1-\lambda_1 I_{r_1}]^{r_1} = 0.$$

Hence $$[A-\lambda_1 I_n]^{r_1}K_1 = K_1[A_1-\lambda_1 I_{r_1}]^{r_1} = 0.$$

Since $K$ is a non-singular matrix the $r_1$ column vectors of $K_1$ are linearly independent.

Similarly, corresponding to each characteristic root $\lambda_i$ of $A$ repeated $r_i$ times, we can find a rectangular matrix $K_i$ with $n$ rows and $r_i$ columns such that

$$AK_i = K_i A_i,$$

where

$$A_i = \begin{bmatrix} \lambda_i & \alpha_{12}^{(i)}, & \alpha_{13}^{(i)}, & & \alpha_{1r_i}^{(i)} \\ & \lambda_i, & \alpha_{23}^{(i)}, & \dots & \\ & & \lambda_i, & \dots & \\ & 0 & & \ddots & \\ & & & & \lambda_i \end{bmatrix} \tag{1.7; 1}$$

Let $T = [\tau_{st}]$ be the square matrix obtained by putting together all the rectangular matrices $K_i$. There will be $n$ columns since $\sum r_i$ is the total number of characteristic roots, namely $n$. Also $T$ is non-singular; for if

$$\sum^{n} \alpha_r[\tau_{sr}] = 0,$$

by multiplying on the left by $[A-\lambda_2 I_n]^{r_2}[A-\lambda_3 I_n]^{r_3}\dots$, we obtain

$$\sum_{r=1}^{r_1} \alpha_r[\tau_{sr}] = 0,$$

and we should have all these $\alpha_r$'s zero, since these vectors are the linearly independent column vectors of $K_1$. Similarly, all the $\alpha_r$'s are zero, and $T$ is non-singular.

We have

$$AT = T\begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & A_3 & \\ & 0 & & \end{bmatrix}$$

whence $T^{-1}AT$ is a matrix

$$T^{-1}AT = D = \begin{bmatrix} A_1 & & & 0 \\ & A_2 & & \\ & & A_3 & \\ 0 & & & \ddots \end{bmatrix} \tag{1.7; 2}$$

$$= \operatorname{diag}(A_1, A_2,\dots, A_r)$$

in which matrices of the type $A_i$ (1.7; 1) occur in positions about the leading diagonal, there being zeros above and below these matrices. The portion of $D$ below the leading diagonal consists entirely of zeros.

One further simplification is possible to obtain the classical canonical form. Each matrix $A_i$ is transformed so that the only non-zero terms are in the leading diagonal, and in the diagonal next above this.

The effect of transforming a matrix by a matrix with unity in each position in the leading diagonal, $\xi$ in the $p$th row $q$th column, and zero in all other positions, is to add $\xi$ times the $p$th column to the $q$th column, and subtract $\xi$ times the $q$th row from the $p$th row. We shall call this a transformation of type $G_{pq}$.

We now proceed to transform the submatrix

$$A_i = \begin{bmatrix} \lambda_i, & \alpha_{12}^{(i)}, & & \alpha_{1r_i}^{(i)} \\ & \lambda_i, & & \\ & & \ddots & \\ & & & \lambda_i \end{bmatrix}$$

to its classical canonical form.

If we put $A_i = \lambda_i I_{r_i} + A_i'$, since $I_{r_i}$ is invariant for transformations, $A_i$ and $A_i'$ will be transformed together. Hence there is no loss of generality if we consider the case $\lambda_i = 0$. The first column in this case consists entirely of zeros.

Consider, then, a matrix $X = [x_{st}]$ of order $r^2$, such that $x_{pq} \neq 0$ only for $p < q$. Then $X^r = 0$. The proof depends upon the equation $X^r = 0$, and transformations may not conserve the zeros on and below the leading diagonal.

Let $X^{p+1}$ be the lowest power of $X$ which is equal to zero. Let the first column of $X^p$ which contains a non-zero element be column $\alpha_p$. If there is more than one non-zero element in this column, suppose that they occur in rows $\alpha_0$, $\alpha_0'$, $\alpha_0''$,.... Then transformations of type $G_{\alpha_0'\alpha_0}$, $G_{\alpha_0''\alpha_0}$,... upon $X$, and therefore upon $X^p$, will reduce all these to zero except in row $\alpha_0$.

From the equation $X.X^p = 0$, it follows that in $X$ the column $\alpha_0$ contains only zeros.

Further, we can arrange so that the element in the column $\alpha_p$ of $X^p$ is the only non-zero element in the $\alpha_0$th row, for if there is another non-zero element, say in column $\alpha_p'$, a transformation of type $G_{\alpha_p\alpha_p'}$ will reduce it to zero. The equation $X^p.X = 0$ now shows that in $X$ the row $\alpha_p$ contains only zeros.

Now consider $X^{p-1}$. The $\alpha_0$th row of this matrix must contain at least one non-zero element since $X^{p-1}.X = X^p$. If the first such element is in column $\alpha_{p-1}$, then, if necessary, transformations of type $G_{\alpha_{p-1}\alpha'_{p-1}}$ will reduce any subsequent element in this row to zero.

Similarly, there is at least one non-zero element in the column $\alpha_p$ of $X^{p-1}$, and if the last such element is in row $\alpha_1$, transformations of type $G_{\alpha'_1\alpha_1}$ will reduce any other such element to zero.

From the equation $X^{p-1}.X = X^p$ we see that, in $X$, row $\alpha_{p-1}$ contains but one non-zero element, namely in column $\alpha_p$, and from $X.X^{p-1} = X^p$ column $\alpha_1$ of $X$ contains but one non-zero element, namely in row $\alpha_0$.

We proceed thus consecutively with $X^{p-2}$, $X^{p-3}$,..., $X^{\frac{1}{2}p}$ or $X^{\frac{1}{2}(p+1)}$. If each of these has but a single non-zero element in the row $\alpha_0$ and in the column $\alpha_p$, these being the only non-zero elements in the respective column and row, then the equations $X^rX^{p-r} = X^p$ show that the lower powers of $X$ have the same property, and $X$ is transformed so that in the rows $\alpha_0$, $\alpha_1$, $\alpha_2$,..., $\alpha_{p-1}$ each has but a single non-zero element, these being respectively in columns $\alpha_1$, $\alpha_2$,..., $\alpha_p$, while the row $\alpha_p$ of $X$ contains only zeros.

A permutation transformation, i.e. one which has the effect of permuting the order of the rows and columns, will bring these rows and columns to the top left-hand corner of the matrix, and $X$ is transformed into the form

$$\begin{array}{cc} X'_1 & \\ & X' \end{array}$$

where $X'_1$ is of the form

$$\begin{array}{cccccc} 0, & p_1, & 0, & 0 & \dots & \\ 0, & 0, & p_2, & 0 & \dots & \\ 0, & 0, & 0, & p_3 & \dots & \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \\ \cdot & \cdot & \cdot & \cdot & \cdot & p_i \\ 0. & . & . & . & . & 0 \end{array}$$

A further transformation of $X'_1$ by the matrix

$$\mathrm{diag}(1, p_1^{-1}, (p_1p_2)^{-1}, (p_1p_2p_3)^{-1}, \dots)$$

will bring the value of each $p_i$ to unity. The matrix $X'$ is then treated in the same manner. Combining the result with (1.7; 2) we obtain

II. *Any matrix* $[a_{st}]$ *may be transformed into the classical canonical form*

$$\begin{bmatrix} A_{11} & & & & & \\ & A_{12} & & & & \\ & & & & & \\ & & & A_{1e_1} & & 0 \\ & & & & A_{21} & \\ & & & & & \\ & & & & & A_{ij} \end{bmatrix}$$

*where each submatrix* $A_{ij}$ *is of the form*

$$A_{ij} = \begin{bmatrix} \lambda_i, & 1, & 0, & 0, & \dots & 0 \\ & \lambda_i, & 1, & 0, & \dots & 0 \\ & & \lambda_i, & 1, & \dots & 0 \\ & & & & & \\ & 0 & & & \lambda_i, & 1 \\ & & & & & \lambda_i \end{bmatrix}$$

*with* $\lambda_i$ *in each position in the leading diagonal, unity in each position in the diagonal immediately above it, and with zeros elsewhere.*

Obviously, two matrices with the same canonical form are equivalent, since they are transforms of the same matrix. Further, we shall show that for a given matrix the canonical form is unique save for the order in which the submatrices $A_{ij}$ are arranged in the leading diagonal. For this it is only necessary to show that, for a given matrix $A_i$ with characteristic equation $(\lambda-\lambda_i)^{q_i} = 0$, the set of numbers which are the orders of the submatrices $A_{ij}$, when $A_i$ is transformed into canonical form, is an invariant function of the matrix.

If $p_i$ is the least index for which

$$(A_i-\lambda_i I)^{p_i} = 0,$$

clearly $p_i^2$ is the order of the largest submatrix.

Again, the rank of $(A_i-\lambda_i I)^{p_i-1}$ is equal to the number of submatrices of order $p_i^2$. The rank of $(A_i-\lambda_i I)^{p_i-2}$ is equal to twice the number of submatrices of order $p_i^2$, plus the number of submatrices of order $(p_i-1)^2$, and so on.

Since the rank of a matrix is unaltered by a transformation, these numbers are invariant properties of the matrix, and it follows that the canonical form is unique.

## 1.8. Various properties of matrices

### The reduced characteristic equation

The equation of least degree satisfied by a matrix $A$ is called the *reduced characteristic equation.* A transform of a matrix clearly satisfies the same reduced equation, hence we may obtain this equation from the canonical form.

Every root of the characteristic equation must necessarily be a root of the reduced equation, and the following theorem is apparent.

I. *The multiplicity of any root in the reduced characteristic equation of a matrix is equal to the number of rows or columns in the largest submatrix corresponding to this root in the canonical form.*

One particular case of this theorem is very important.

II. *If the reduced characteristic equation of a matrix has no multiple roots, the canonical form is diagonal.*

The converse is also true, but obvious.

### The spur of a matrix

The sum of the elements in the leading diagonal of a matrix $[a_{st}]$, namely $\sum a_{ii}$, is called the *spur* of the matrix. It is, with a minus sign attached, the first coefficient in the characteristic equation of the matrix, and is equal to the sum of the characteristic roots.

III. *The spurs of equivalent matrices are equal.*

### Nilpotent matrices

If a matrix $A$ is such that some power of it is zero, i.e.

$$A^n = 0,$$

it is said to be *nilpotent.*

Clearly all the characteristic roots of a nilpotent matrix are zero.

IV. *The spur of a nilpotent matrix is zero.*

### Idempotent matrices

A matrix $A$ is said to be *idempotent* if it satisfies $A^2 = A$. The reduced equation has no multiple roots, and hence the canonical form of an idempotent matrix is diagonal. We can find $T$ such that

$$T^{-1}AT = \mathrm{diag}(1^r, 0^{n-r}).$$

V. *The rank of an idempotent matrix is equal to its spur.*

VI. *If $A_1$, $A_2,\dots, A_p$ are idempotent matrices such that the product of any two is zero, they may be transformed simultaneously into diagonal form.*

Let $A_1$ be transformed into the form $\operatorname{diag}(1^{a_1}, 0^{n-a_1})$. Then since $A_1 A_i = A_i A_1 = 0$, $A_i$ contains only zero elements in the first $a_1$ rows and columns. A transformation not affecting these rows and columns will bring $A_2$ into diagonal form, and so on.

## 1.9. Unitary and orthogonal matrices

Let $[a_{st}]$ be the matrix of a transformation

$$[x'_s] = [a_{st}][x_s].$$

Denote the complex conjugate of a quantity by a bar. Then, if $[a_{st}]$ leaves the form

$$[\bar{x}_t][x_s] = \sum x_i \bar{x}_i$$

invariant, $[a_{st}]$ is called a *unitary matrix.*

Since $$[x'_s] = [a_{st}][x_s],$$
we have also $$[x'_t] = [x_t][a_{ts}]$$
and thus $$[\bar{x}'_t] = [\bar{x}_t][\bar{a}_{ts}].$$
Hence
$$\begin{aligned}[\bar{x}_t][x_s] &= [\bar{x}'_t][x'_s] \\ &= [\bar{x}_t][\bar{a}_{ts}][a_{st}][x_s].\end{aligned}$$

It follows that the necessary and sufficient condition that a matrix $[a_{st}]$ should be unitary, is that

$$[\bar{a}_{ts}][a_{st}] = I = [\delta_{st}]. \tag{1.9; 1}$$

These conditions may be written

$$\left.\begin{aligned}\sum_r a_{rp}\,\bar{a}_{rp} &= 1, \\ \sum_r a_{rp}\,\bar{a}_{rq} &= 0 \quad (p \neq q).\end{aligned}\right\} \tag{1.9; 2}$$

An alternative set of conditions obtained from the equation $[a_{st}][\bar{a}_{ts}] = I$ is

$$\left.\begin{aligned}\sum_r a_{pr}\,\bar{a}_{pr} &= 1, \\ \sum_r a_{pr}\,\bar{a}_{qr} &= 0 \quad (p \neq q).\end{aligned}\right\} \tag{1.9; 3}$$

I. *The product of two unitary matrices is unitary.*

This follows immediately from the definition.

II. *The characteristic roots of a unitary matrix have modulus unity.*

Let $\lambda_1$ be a characteristic root of the unitary matrix $[a_{st}]$ and let

$$[a_{st}][x_s] = \lambda_1[x_s].$$

Then we have also $$[\bar{x}_t][\bar{a}_{ts}] = \bar{\lambda}_1[\bar{x}_t],$$

so that $$\begin{aligned}[\bar{x}_t][x_s] &= [\bar{x}_t][\bar{a}_{ts}][a_{st}][x_s] \\ &= \lambda_1\bar{\lambda}_1[\bar{x}_t][x_s].\end{aligned}$$

Since $[\bar{x}_t][x_s]$ is an essentially positive form, it follows that $\lambda_1\bar{\lambda}_1 = 1$, and $\lambda_1$ has unit modulus.

III. *A unitary matrix may be transformed into diagonal form by another unitary matrix.*

The proof is by induction. We will assume the theorem to be true for matrices of order $(n-1)^2$ and prove it true for matrices of order $n^2$.

Let $\lambda_1$ be a characteristic root of the unitary matrix $[a_{st}]$ of order $n^2$, and let $$[a_{st}][x_{s1}] = \lambda_1[x_{s1}].$$

The vector $[x_{s1}]$ may clearly be chosen so that

$$\sum x_{r1}\bar{x}_{r1} = 1.$$

Further, a unitary matrix $[x_{st}]$ may be found such that the first column is $[x_{s1}]$. The columns of such a matrix may be chosen consecutively to satisfy equations (1.9; 2). In order to satisfy the equations $$\sum x_{rp}\bar{x}_{rq} = 0,$$

we must find a solution in each case of a set of less than $n$ homogeneous linear equations in $n$ variables, which is always possible. The whole column is then multiplied by a suitable numerical factor to ensure that $$\sum_r x_{rp}\bar{x}_{rp} = 1.$$

If $[x_{st}]$ is thus chosen, then the first column of the product $[a_{st}][x_{st}]$ is equal to $[\lambda_1 x_{s1}]$. It follows that the first column of

$$[x_{st}]^{-1}[a_{st}][x_{st}] = [b_{st}]$$

is given by $$b_{11} = \lambda_1,$$

$$b_{r1} = 0 \quad (r \neq 1).$$

But $[b_{st}]$ is unitary, and thus

$$\sum b_{r1}\bar{b}_{rp} = 0 \quad (p \neq 1),$$

so that $$\lambda_1\bar{b}_{1p} = 0 \quad (p \neq 1)$$

and $$b_{1r} = b_{r1} = 0 \quad (r \neq 1).$$

The matrix $[b_{st}]$ is thus of the form

$$\left.\begin{array}{c|c} \lambda_1 & 0 \\ \hline 0 & [b'_{st}] \end{array}\right.$$

Since $[b'_{st}]$ is a unitary matrix of order $(n-1)^2$ it can be transformed into diagonal form by another unitary matrix, and since the product of two unitary matrices is unitary, the theorem follows.

A matrix $[a_{st}]$ is *orthogonal* if it leaves the form $[x_t][x_s] = \sum x_i^2$ invariant.

If the elements $a_{pq}$ are all real, we may choose the vector $[x_s]$ to be real also, and thus $\sum x_i \bar{x}_i = \sum x_i^2$.

IV. *A real unitary matrix is a real orthogonal matrix, and conversely.*

The necessary and sufficient conditions that a real matrix $[a_{st}]$ should be orthogonal are

$$\left.\begin{aligned} \sum_r a_{pr}^2 &= 1, \\ \sum_r a_{pr} a_{qr} &= 0 \quad (p \neq q), \end{aligned}\right\}$$

or, alternatively,

$$\left.\begin{aligned} \sum_r a_{rp}^2 &= 1, \\ \sum_r a_{rp} a_{rq} &= 0 \quad (p \neq q). \end{aligned}\right\}$$

From Theorem II we have

V. *The characteristic roots of a real orthogonal matrix are either* $\pm 1$, *or in pairs* $e^{i\theta}$, $e^{-i\theta}$, *with* $\theta$ *real.*

Subsequently, by orthogonal matrix we shall mean a real orthogonal matrix.

An orthogonal matrix cannot in general be transformed into diagonal form by another orthogonal matrix, for the characteristic roots are not, in general, real, but it can be so transformed by a unitary matrix.

Let $[a_{st}]$ be an orthogonal matrix of order $n^2$, and let

$$[x_{st}] \equiv [x'_{st} + i x''_{st}]$$

be a unitary matrix which transforms $[a_{st}]$ into diagonal form, so that

$$[x_{st}]^{-1}[a_{st}][x_{st}] = \operatorname{diag}(\alpha_1,\ldots, \alpha_n) = D.$$

There are four cases. If $n = 2\nu+1$, and $|a_{st}| = +1$, we may put

$$D = \operatorname{diag}(1, e^{i\phi_1}, e^{-i\phi_1},\ldots, e^{i\phi_\nu}, e^{-i\phi_\nu}),$$

and for $|a_{st}| = -1$,

$$D = \operatorname{diag}(-1, e^{i\phi_1}, e^{-i\phi_1},..., e^{i\phi_\nu}, e^{-i\phi_\nu}).$$

If $n = 2\nu$, we may put, for $|a_{st}| = +1$,

$$D = \operatorname{diag}(e^{i\phi_1}, e^{-i\phi_1},..., e^{i\phi_\nu}, e^{-i\phi_\nu}),$$

and for $|a_{st}| = -1$,

$$D = \operatorname{diag}(1, -1, e^{i\psi_1}, e^{-i\psi_1},..., e^{i\psi_{\nu-1}}, e^{-i\psi_{\nu-1}}).$$

Corresponding to the characteristic root $e^{i\phi_r}$ there is a column $[x_{sr}] = [x'_{sr}+ix''_{sr}]$ of the matrix $[x_{st}]$ such that

$$[a_{st}][x'_{sr}+ix''_{sr}] = e^{i\phi_r}[x'_{sr}+ix''_{sr}].$$

Equating real and imaginary parts,

$$[a_{st}][x'_{sr}] = [x'_{sr}\cos\phi_r - x''_{sr}\sin\phi_r],$$
$$[a_{st}][x''_{sr}] = [x'_{sr}\sin\phi_r + x''_{sr}\cos\phi_r].$$

Clearly $[x'_{sr}-ix''_{sr}]$ may be taken as a column of the matrix $[x_{st}]$ corresponding to the characteristic root $e^{-i\phi_r}$.

If these two columns are replaced by the columns $[\sqrt{2}\,x'_{sr}]$, $[-\sqrt{2}\,x''_{sr}]$, and every pair of complex columns of $[x_{st}]$ is similarly replaced, we obtain a real orthogonal matrix.

VI. *Any orthogonal matrix may be transformed by another orthogonal matrix into one of the four forms*

$$\operatorname{diag}(1, \Phi_1, \Phi_2,..., \Phi_\nu),$$
$$\operatorname{diag}(-1, \Phi_1, \Phi_2,..., \Phi_\nu),$$
$$\operatorname{diag}(\Phi_1,..., \Phi_\nu),$$
$$\operatorname{diag}(1, -1, \Psi_1,..., \Psi_{\nu-1}),$$

*where*

$$\Phi_r = \begin{bmatrix} \cos\phi_r, & \sin\phi_r \\ -\sin\phi_r, & \cos\phi_r \end{bmatrix},$$

$$\Psi_r = \begin{bmatrix} \cos\psi_r, & \sin\psi_r \\ -\sin\psi_r, & \cos\psi_r \end{bmatrix}.$$

The determinant of an orthogonal matrix is clearly $\pm 1$. A linear transformation of which the matrix is an orthogonal matrix of positive determinant is called a *rotation*.

### Hermitian and skew-Hermitian matrices

A matrix $[a_{st}]$ which satisfies

$$[a_{st}] = [\bar{a}_{ts}],$$

the bar denoting the complex conjugate, so that

$$a_{pq} = \bar{a}_{qp},$$

and in particular $a_{pp}$ is real, is called a *Hermitian* matrix.

A matrix $[a_{st}]$ which satisfies

$$[a_{st}] = -[\bar{a}_{ts}]$$

so that

$$a_{pq} = -\bar{a}_{qp},$$

and in particular $a_{pp}$ is a pure imaginary, is called a *skew-Hermitian* matrix.

The property of being Hermitian or skew-Hermitian is in each case unaltered in transformation by a unitary matrix. Let $[b_{st}]$ be unitary. Then the transpose of

$$[c_{st}] = [b_{st}]^{-1}[a_{st}][b_{st}]$$

is clearly

$$[b_{ts}][a_{ts}][b_{ts}]^{-1} = [\bar{b}_{st}]^{-1}[a_{ts}][\bar{b}_{st}].$$

If $[a_{st}]$ is Hermitian, this is clearly the complex conjugate of $[c_{st}]$, and if $[a_{st}]$ is skew-Hermitian, it is equal to $-[\bar{c}_{st}]$.

The corresponding real matrices are *symmetric* matrices which satisfy

$$[a_{ts}] = [a_{st}]$$

and *skew-symmetric* matrices which satisfy

$$[a_{ts}] = -[a_{st}].$$

The properties of being symmetric or skew-symmetric are unaltered in transformations by orthogonal matrices.

Skew-Hermitian and skew-symmetric matrices are closely connected respectively with unitary matrices, and with orthogonal matrices.

If $A$ is a skew-Hermitian matrix, it is easily verified that $[I+A][I-A]^{-1}$ is unitary. Conversely, if a unitary matrix $B$ is given, a skew-Hermitian matrix $A$ can be found such that

$$B = [I+A][I-A]^{-1},$$

provided that $-1$ is not a characteristic root of $B$.

The same is true if $A$ is skew-symmetric, and $B$ is orthogonal. Again, $-1$ must not be a characteristic root of $B$.

### Infinitesimal unitary transformations and infinitesimal rotations

If the matrix of a transformation is of the form $I+[\epsilon_{st}]$, where a limiting process is to be taken in which each element $\epsilon_{ij}$ of $[\epsilon_{st}]$ tends to zero, so that squares and products of these elements may be ignored, the transformation is said to be *infinitesimal.*

Clearly, if $I+[\epsilon_{st}]$ is the matrix of an infinitesimal unitary transformation, then $[\epsilon_{st}]$ is skew-Hermitian. Conversely, if $I+[\epsilon_{st}]$ is the matrix of an infinitesimal transformation, and $[\epsilon_{st}]$ is skew-Hermitian, then $I+[\epsilon_{st}]$ is unitary.

Again, if $I+[\epsilon_{st}]$ is the matrix of an infinitesimal orthogonal transformation, $[\epsilon_{st}]$ is skew-symmetric, and if $I+[\epsilon_{st}]$ is the matrix of an infinitesimal transformation and $[\epsilon_{st}]$ is skew-symmetric, then $I+[\epsilon_{st}]$ is orthogonal.

Let $[a_{st}]$ be any unitary matrix, and let $[b_{st}]$ be another unitary matrix which transforms it into diagonal form so that

$$[b_{st}]^{-1}[a_{st}][b_{st}] = \mathrm{diag}(e^{i\theta_1}, e^{i\theta_2},\ldots, e^{i\theta_n}).$$

Put $$\Theta = \mathrm{diag}(i\theta_1, i\theta_2,\ldots, i\theta_n),$$

and put $$\Theta' = [b_{st}]\Theta[b_{st}]^{-1}.$$

Clearly $\Theta$ and $\Theta'$ are skew-Hermitian.

We have readily

$$\lim_{n\to\infty}\left[I+\frac{1}{n}\Theta\right]^n = \mathrm{diag}(e^{i\theta_1},\ldots, e^{i\theta_n}),$$

and thus

$$\lim_{n\to\infty}\left[I+\frac{1}{n}\Theta'\right]^n = [b_{st}]\lim_{n\to\infty}\left[I+\frac{1}{n}\Theta\right]^n[b_{st}]^{-1} = [a_{st}].$$

VII. *Any unitary matrix $[a_{st}]$ may be expressed in the form*

$$\lim_{n\to\infty}\left[I+\frac{1}{n}\Theta\right]^n,$$

*where $\Theta$ is a skew-Hermitian matrix.*

With orthogonal matrices a similar result is only possible if the determinant of the matrix is $+1$.

We transform the orthogonal matrix $[a_{st}]$ by an orthogonal matrix $[b_{st}]$ into one of the two forms

$$\mathrm{diag}(1, \Phi_1, \Phi_2,\ldots, \Phi_\nu),$$

or $$\mathrm{diag}(\Phi_1, \Phi_2,\ldots, \Phi_\nu),$$

where $$\Phi_r = \begin{bmatrix} \cos\phi_r, & \sin\phi_r \\ -\sin\phi_r, & \cos\phi_r \end{bmatrix}.$$

Put $$\Phi_r' = \begin{bmatrix} 0, & \phi_r \\ -\phi_r, & 0 \end{bmatrix}.$$

Then clearly $$\Phi_r = \lim_{n\to\infty}\left[I+\frac{1}{n}\Phi_r'\right]^n.$$

Put $\Phi' = \text{diag}(0, \Phi'_1, \ldots, \Phi'_\nu)$ or $\text{diag}(\Phi'_1, \ldots, \Phi'_\nu)$,

and put $\Phi = [b_{st}]\Phi'[b_{st}]^{-1}$.

We obtain

VIII. *An orthogonal matrix* $[a_{st}]$ *of positive determinant may be expressed in the form* $\lim_{n\to\infty}\left[I+\frac{1}{n}\Phi\right]^n$, *where* $\Phi$ *is a skew-symmetric matrix.*

An infinitesimal orthogonal transformation is called an *infinitesimal rotation.* An orthogonal transformation that can be generated by infinitesimal rotations, i.e. an orthogonal transformation of positive determinant, is called a *rotation.*

### The direct product of two matrices

If $[a_{st}]$ and $[b_{st}]$ are square matrices of orders $m^2$ and $n^2$ respectively, then the square matrix of order $m^2n^2$

$$[a_{st}b_{s't'}]$$

in which the pair of numbers $s$, $s'$ define the row, and $t$, $t'$ the column, is called the direct product of the matrices $[a_{st}]$, $[b_{st}]$ and is denoted by $[a_{st}]\times[b_{st}]$. Any convention may be adopted for the ordering of the rows in a direct product, provided that the same convention is adopted for the columns. A suitable convention is to take the pairs of numbers $(s, s')$, and similarly $(t, t')$ in 'dictionary' order.

If the square matrices $A$ and $C$ are of order $m^2$, and $B$ and $D$ of order $n^2$, then

$$(A\times B)(C\times D) = AC\times BD.$$

This equation is implicit in the definition of a direct product. The direct product of two groups or of two algebras is defined so that the elements satisfy this equation, each element in the direct product corresponding to a pair of elements in the respective groups or algebras.

It may be verified that in the direct product of two matrices the spur is the product of the respective spurs of the two matrices, and the characteristic roots of the direct product are the products, in pairs, of the characteristic roots of the original matrices.

# II

# ALGEBRAS

## 2.1. Definition of an algebra over the complex numbers

CHOOSE any $n^3$ complex numbers $\gamma_{ijk}$ $(1 \leqslant i, j, k \leqslant n)$, and consider all $n$-tuples $x \equiv [\xi_1, \xi_2,..., \xi_n]$ of $n$ ordered complex numbers. Then if addition, multiplication, and scalar multiplication are defined according to the following rules

$$\left.\begin{aligned} [\xi_1,\xi_2,...,\xi_n]+[\eta_1,\eta_2,...,\eta_n] &= [\xi_1+\eta_1,\xi_2+\eta_2,...,\xi_n+\eta_n], \\ [\xi_1,\xi_2,...,\xi_n]\cdot[\eta_1,\eta_2,...,\eta_n] & \\ = [\textstyle\sum \gamma_{ij1}\xi_i\eta_j, \sum \gamma_{ij2}\xi_i\eta_j,..., \sum \gamma_{ijn}\xi_i\eta_j], & \\ \lambda[\xi_1,\xi_2,...,\xi_n] = [\lambda\xi_1,\lambda\xi_2,...,\lambda\xi_n], & \end{aligned}\right\} \quad (2.1;1)$$

$\lambda$ being a scalar, i.e. a complex number, the $n$-tuples are said to form an *algebra of order n over the complex numbers*. The $n$-tuples $x$ are called *elements* of the algebra.

More generally, the quantities $\xi_i$, $\gamma_{ijk}$ may be taken from any specified field, when the corresponding $n$-tuples are said to form an algebra over this field. For the general definition of a field, and for a very thorough account of algebras over a general field, the reader is referred to L. E. Dickson's *Algebras and their Arithmetics* (Chicago, 1923).

If for any three elements $x$, $y$, $z$ of an algebra we have

$$(xy)z = x(yz),$$

the algebra is said to be *associative*. The condition that an algebra is associative is clearly that

$$\sum_p \gamma_{ijp}\gamma_{pkr} = \sum_p \gamma_{jkp}\gamma_{ipr}, \qquad (2.1;2)$$

for all $i$, $j$, $k$, and $r$.

Comparatively little is known of non-associative algebras, and it will be assumed that all the algebras considered here are associative.

If

$$e_1 = [1,0,...,0], \quad e_2 = [0,1,0,...,0], \quad ..., \quad e_n = [0,...,0,1],$$

then any element $x$ may be expressed in the form $x = \sum \xi_i e_i$, and the multiplication rule takes the form

$$e_i e_j = \sum \gamma_{ijk} e_k.$$

The elements $e_1$, $e_2$,..., $e_n$ are said to form a *basis* of the algebra.

## 2.2. Change of basis and the regular matrix representation of an algebra

If $\epsilon_1, \epsilon_2,\ldots, \epsilon_n$ are $n$ linearly independent (over the complex numbers) elements of an algebra of order $n$, then any element $x$ of the algebra may be expressed in the form

$$x = \sum \xi'_r \epsilon_r.$$

Let

$$\epsilon_i \epsilon_j = \sum \Gamma_{ijk} \epsilon_k.$$

Then we may define an algebra of the $n$-tuples $(\xi'_1, \xi'_2,\ldots, \xi'_n)$ with constants of multiplication $\Gamma_{ijk}$, which is simply isomorphic with the original algebra, i.e. corresponding to elements $x$, $y$,... of the original algebra there are elements $x'$, $y'$,... of the new algebra such that $x'+y'$ corresponds to $x+y$, and $x'y'$ to $xy$, and conversely.

These are called *equivalent algebras*. Without reference to the individual numbers in an $n$-tuple, there is no means of distinguishing between equivalent algebras, and they may be regarded as the same algebra.

The elements $\epsilon_1, \epsilon_2,\ldots, \epsilon_n$ are said to form a basis of the original algebra, and the change from the expression of $x$ as $\sum \xi_i e_i$, or by the $n$-tuple $[\xi_1,\ldots, \xi_n]$, to the form $\sum \xi'_i \epsilon_i$ or $[\xi'_1,\ldots, \xi'_n]$ is called a *change of basis*.

**The regular matrix representation.** (See also p. viii)

Corresponding to the element $x = \sum \xi_r e_r$ form the matrix

$$X = [\sum \xi_r \gamma_{srt}].$$

Then the matrix corresponding to a second element $y = \sum \eta_r e_r$ is

$$Y = [\sum \eta_r \gamma_{srt}].$$

Hence the product $XY$ of the matrices corresponds to the product of the elements, $xy = \sum \xi_r \eta_p \gamma_{rpu} e_u$, for

$$\begin{aligned} XY &= [\sum \xi_r \eta_p \gamma_{sru} \gamma_{upt}] \\ &= [\sum \xi_r \eta_p \gamma_{rpu} \gamma_{sut}], \end{aligned}$$

since from the associative condition (2.1; 2)

$$\sum \gamma_{sru} \gamma_{upt} = \sum \gamma_{rpu} \gamma_{sut}.$$

The set of matrices is simply isomorphic with the set of elements of the algebra, and is said to give a *matrix representation of the algebra*.

This representation is called the *regular matrix representation of the algebra*.

There may be other matrix representations. For example, the set of all matrices of order $n^2$ clearly form an algebra of order $n^2$, and the matrices themselves form a matrix representation of the algebra. But the *regular* matrix representation is of order $n^4$, and is clearly a distinct representation.

For a different basis of the algebra, we obtain a different regular matrix representation, but we shall show that the effect of a change of basis is merely to effect a transformation on the matrices $X$, $Y$

Let the basis be changed from $e_1, \dots, e_n$ to $\epsilon_1, \dots, \epsilon_n$, where

$$\epsilon_p = \sum \alpha_{pq} e_q,$$

$$e_q = \sum \beta_{qp} \epsilon_p,$$

so that

$$[\alpha_{st}][\beta_{st}] = I.$$

Let the multiplication rules relative to the two bases be respectively

$$e_p e_q = \sum \gamma_{pqr} e_r,$$

$$\epsilon_p \epsilon_q = \sum \Gamma_{pqr} \epsilon_r.$$

Then

$$\begin{aligned} \sum \Gamma_{uvw} \epsilon_w &= \epsilon_u \epsilon_v \\ &= \sum \alpha_{up} \alpha_{vq} e_p e_q \\ &= \sum \gamma_{pqr} \alpha_{up} \alpha_{vq} e_r \\ &= \sum \gamma_{pqr} \alpha_{up} \alpha_{vq} \beta_{rw} \epsilon_w, \end{aligned}$$

so that

$$\Gamma_{uvw} = \sum \gamma_{pqr} \alpha_{up} \alpha_{vq} \beta_{rw}.$$

Now let the regular matrix representations of $x = \sum \xi_r e_r = \sum \xi'_r \epsilon_r$ corresponding to the two bases be $X$ and $X'$ respectively.

We have

$$x = \sum \xi'_r \epsilon_r = \sum \xi'_r \alpha_{rp} e_p,$$

and hence

$$\xi_p = \sum \alpha_{rp} \xi'_r.$$

Thus

$$\begin{aligned} X' &= [\sum \xi'_r \Gamma_{srt}] \\ &= [\sum \xi'_r \gamma_{uvw} \alpha_{su} \alpha_{rv} \beta_{wt}] \\ &= [\sum \xi_v \gamma_{uvw} \alpha_{su} \beta_{wt}] \\ &= [\alpha_{st}][\sum \xi_v \gamma_{svt}][\beta_{st}] \\ &= [\beta_{st}]^{-1} X [\beta_{st}], \end{aligned}$$

so that $X'$ is a transform of $X$.

The characteristic equation of $X$ is invariant for transformations, and is thus independent of the basis of the algebra. It is called the *characteristic equation of the element $x$ of the algebra.*

## 2.3. Simple matrix algebras

If in an algebra of order $n^2$ a basis $e_{ij}$ $(1 \leqslant i, j \leqslant n)$ can be found such that

$$e_{ij}\,e_{jk} = e_{ik},$$

$$e_{ij}\,e_{kp} = 0 \quad (j \neq k),$$

the algebra is called a *simple matrix algebra*. It is obviously equivalent to the algebra of matrices of order $n^2$, the element $x = \sum x_{ij}\,e_{ij}$ corresponding to the matrix $[x_{st}]$.

There are an infinity of ways of expressing the algebra as a simple matrix algebra, for if $P$ is any fixed matrix of order $n^2$, the element $x$ may equally well be made to correspond to the matrix $P^{-1}[x_{st}]P$. We shall show that the *spur* of the matrix $[x_{st}]$ is independent of the mode of representation as a simple matrix algebra.

## 2.4. Examples of associative algebras

(*a*) Let the multiplication table of an algebra of order 4 be

$$\begin{array}{llll} e_1^2 = e_1, & e_1\,e_2 = e_2, & e_1\,e_3 = 0, & e_1\,e_4 = 0, \\ e_2\,e_1 = 0, & e_2\,e_2 = 0, & e_2\,e_3 = e_1, & e_2\,e_4 = e_2, \\ e_3\,e_1 = e_3, & e_3\,e_2 = e_4, & e_3\,e_3 = 0, & e_3\,e_4 = 0, \\ e_4\,e_1 = 0, & e_4\,e_2 = 0, & e_4\,e_3 = e_3, & e_4\,e_4 = e_4. \end{array}$$

This algebra is a simple matrix algebra of order 4, the element $x = \sum \xi_r\,e_r$ corresponding to the matrix

$$\begin{bmatrix} \xi_1, & \xi_2 \\ \xi_3, & \xi_4 \end{bmatrix}.$$

(*b*) If the basis of the above algebra is changed to

$$I = e_1+e_4, \qquad i = \surd(-1)e_1-\surd(-1)e_4,$$

$$j = e_2-e_3, \qquad k = \surd(-1)e_2+\surd(-1)e_3,$$

the multiplication table becomes

$$Ii = iI = i, \qquad Ij = jI = j, \qquad Ik = kI = k,$$

$$ij = -ji = k, \qquad jk = -kj = i, \qquad ki = -ik = j,$$

$$i^2 = j^2 = k^2 = -I, \qquad I^2 = I.$$

This is the familiar multiplication table of Hamilton's *Quaternions*.

(*c*) There is an associative algebra of order 2 with multiplication table

$$e_1^2 = e_1, \qquad e_1\,e_2 = \qquad\qquad e_2^2 = e_2\,e_1 = 0.$$

## 2.5. Linear sets and sub-algebras

Let $A$ denote the set of all the elements of an algebra. We shall refer to the *algebra* $A$.

If $x_1, x_2,..., x_r$ are any elements of the algebra $A$, the set of elements

$$\alpha_1 x_1 + \alpha_2 x_2 + ... + \alpha_r x_r,$$

where $\alpha_1,..., \alpha_r$ are any complex numbers, is said to form a *linear set*. The least number of elements $x_i$ necessary to define the set is called the *order* of the set.

The *sum* of two sets $B_1$ and $B_2$, denoted by $B_1 \dot{+} B_2$, is defined as the set of elements linearly dependent upon the two sets of elements $B_1$ and $B_2$.

The *meet* of two sets $B_1$ and $B_2$, denoted by $B_1 \wedge B_2$, is the set of elements which belong to both sets.

Clearly the sum and meet of two linear sets are also linear sets.

If $z_1$ is a fixed element of the algebra, the set of elements $z_1 x$, where $x$ is a variable element of the algebra $A$, is a linear set which may or may not coincide with $A$. It is denoted by $z_1 A$. The linear sets $Az_2$ and $z_1 A z_2$ are similarly defined.

If the product of any two elements of a linear set $B$ of an algebra $A$ belongs to the set $B$, then the set $B$ clearly forms an algebra, which is called a *sub-algebra* of $A$.

If an algebra $A$ has a sub-algebra $B$ such that if $a$ and $b$ are any two elements of $A$ and $B$ respectively, then $ab$ and $ba$ both belong to $B$, then $B$ is called an *invariant sub-algebra* of $A$. This condition may be otherwise expressed; $xBy$ is included in $B$ for all $x$ and $y$ belonging to $A$, or $ABA$ is included in $B$.

If an algebra $A$ has two invariant sub-algebras $B_1$ and $B_2$ such that $B_1 \wedge B_2 = 0$, $B_1 \dot{+} B_2 = A$, then $A$ is said to be *reducible*, and to be the *direct sum* of the algebras $B_1$ and $B_2$. Any element $x$ of $A$ may be expressed in the form $x = x_1 + x_2$, where $x_1$ belongs to $B_1$ and $x_2$ to $B_2$, and if $y$ is expressed similarly as $y = y_1 + y_2$, then

$$xy = x_1 y_1 + x_2 y_2.$$

For the cross-products $x_1 y_2$ and $x_2 y_1$ belong to both sub-algebras, and hence to their meet, which is zero.

## 2.6. Modulus, idempotent, and nilpotent elements

If an algebra possesses an element $m$ such that $xm = mx = x$ for every element $x$ of the algebra, $m$ is called the *modulus* of the algebra, and the algebra is said to *possess a modulus*.

The modulus, if it exists, is clearly unique, for if $m$ and $s$ are moduli, then $ms = s$ and $ms = m$, so that $s = m$. The algebra ($a$) in the examples in § 2.4 has the modulus $e_1+e_4$. The algebra ($c$) does not possess a modulus.

If an algebra possesses a modulus $m$, the set of elements $\xi m$, where $\xi$ is a scalar, is simply isomorphic with the complex numbers. Multiplication by $\xi m$ is the same as scalar multiplication by $\xi$. We may omit the $m$ and treat the elements $\xi m$ as complex numbers.

An element $x$ is *nilpotent* if we can find an index $n$ such that

$$x^n = 0.$$

An element $x$ is *properly nilpotent* in an algebra $A$ if $xy$ and therefore also $yx$ is nilpotent for every element $y$ in $A$.

In the algebra ($a$) of § 2.4, $e_2$ is nilpotent, but not properly nilpotent, since $e_3 e_2 = e_4$, and $e_4 = e_4^2$ is not nilpotent. In the algebra ($c$), $e_2$ is properly nilpotent.

There is a class of algebras which possess no properly nilpotent elements, and it is with these algebras that we are chiefly concerned in this book.

A non-zero element $x$ is said to be *idempotent* (*an idempotent*) if it satisfies $x^2 = x$.

An idempotent $e$ of an algebra $A$ is called a *principal idempotent* if there does not exist an idempotent $u$ such that $eu = ue = 0$. If the algebra possesses a modulus $m$, this is clearly the one and only principal idempotent. For if $e$ is idempotent, $m-e$ also is idempotent, and $e(m-e) = (m-e)e = 0$.

*Any algebra with one idempotent possesses a principal idempotent.*

For if $e$ is idempotent, but not principal, there is an idempotent $u$ such that $eu = ue = 0$. Then $e+u$ is idempotent. If it is not principal, then there is another idempotent $u'$ such that

$$(e+u)u' = u'(e+u) = 0.$$

Continuing thus we must reach a principal idempotent by a finite number of steps, the order of the algebra being finite.

In the case of an algebra without a modulus the principal idempotent may not be unique. Thus for the algebra ($c$) in § 2.4, $e_1$ is a principal idempotent, and so also is $e_1+\xi e_2$ for any complex number $\xi$.

## 2.7. The reduced characteristic equation

By comparison with the regular matrix representation of an algebra it is clear that in general every element $x$ satisfies its characteristic equation $F(x) = 0$, the independent term being taken as a multiple of the modulus. In the case when the algebra does not possess a modulus, zero is a characteristic root of every element $x$ and there is no independent term.

It may happen that $x$ satisfies an equation of lower degree. The equation of least degree with scalar coefficients satisfied by an element $x$ of an algebra is called the *reduced characteristic equation* of $x$. In this equation there may be an independent term involving the modulus. If there is no modulus, $x$ is a factor of the equation.

I. *If the reduced characteristic equation of an element $x$ of an algebra has $p$ distinct roots, and the algebra has a modulus, and in any case if it has $p$ distinct roots other than zero, then there are $p$ linearly independent polynomials in $x$ which are idempotent elements of the algebra.*

Let the reduced characteristic equation of $x$ be

$$(x-\alpha_1)^{a_1}(x-\alpha_2)^{a_2}\ldots(x-\alpha_p)^{a_p} = 0.$$

Put $z_1 = (x-\alpha_2)^{a_2}\ldots(x-\alpha_p)^{a_p}$, and expanding $z_1^{-1}$ formally in ascending powers of $(x-\alpha_1)$, let

$$y_1 = c_0+c_1(x-\alpha_1)+c_2(x-\alpha_1)^2+\ldots+c_{a_1-1}(x-\alpha_1)^{a_1-1}$$

be the first $a_1$ terms.

Then $$y_1 z_1 = 1+(x-\alpha_1)^{a_1}\phi(x),$$

where $\phi(x)$ is a polynomial in $x$.

But $$z_1(x-\alpha_1)^{a_1} = 0,$$

and hence $$(y_1 z_1)^2 = y_1 z_1.$$

Thus $e_1 = y_1 z_1$ is an idempotent corresponding to the root $\alpha_1$.

Corresponding to the roots $\alpha_2,\ldots, \alpha_p$ we obtain in a similar manner idempotents $e_2,\ldots, e_p$. These idempotents are linearly independent, for, since

$$z_i z_j = 0,$$

we have $$e_i e_j = 0 \quad (i \neq j).$$

If the algebra has a modulus, the independent term in such expressions is replaced by a multiple of the modulus. If the algebra has no modulus, $x$ is a factor of the reduced equation, and the idempotents corresponding to the $p$ non-zero roots each contain a factor $x$, and exist as numbers of the algebra. The theorem is true in either case.

II. *If an algebra has no idempotent every element is nilpotent, and hence properly nilpotent.*

III. *An algebra without a modulus contains properly nilpotent elements.*

If the algebra has an idempotent it has a principal idempotent $e$. Since $e$ is not a modulus, either the set of elements $x-ex$ or the set $x-xe$ is distinct from zero. Such a set clearly forms an invariant sub-algebra with no idempotent, since $e$ is a principal idempotent. Elements of the set are properly nilpotent.

## 2.8. Reduction of an algebra relative to an idempotent

Let $e$ be an idempotent of an algebra $A$. Then the linear set $eAe$ is clearly a sub-algebra. Further, $e$ is a modulus for this sub-algebra, for any element $x$ may be put in the form $x = ex'$, so that

$$ex = eex' = ex' = x.$$

Similarly, $xe = x$.

If an idempotent $e$ is expressible in the form $e = e_1+e_2$, where $e_1$ and $e_2$ are idempotents satisfying $e_1 e_2 = e_2 e_1 = 0$, then $e$ is said to be *reducible.* Otherwise it is said to be *irreducible.*

*An idempotent can be expressed as the sum of irreducible idempotents such that the product of any two is zero.* For if it is reducible it is expressible as the sum of two idempotents with products zero. If either of these is reducible it is expressible as the sum of two, and so on. The process must end since the order of the algebra is finite.

*If an algebra $A$ possesses a modulus which is an irreducible idempotent, it is either of order* 1, *or it possesses a properly nilpotent element.*

It possesses no other idempotent $e$, for otherwise $m = e+(m-e)$ would be reducible. Hence the reduced characteristic equation of every element has only one root. If the order of the algebra is greater than 1, let $x$ be any element that is not a scalar multiple of the modulus. Then $x$ satisfies an equation of the form

$$(x-\lambda m)^r = 0.$$

Thus $y = x-\lambda m$ is nilpotent. The linear set $yA$ is a sub-algebra which does not contain the modulus, and hence has no idempotent. Every element of $yA$ is properly nilpotent.

Again, if $x$ is properly nilpotent in $eAe$, where $e$ is an idempotent, it is also properly nilpotent in $A$. For, since $e$ is a modulus for $eAe$, $x = exe$, and

$$(xy)^n = x(eyex)^{n-1}y.$$

Hence if $eyex$ is nilpotent, so also is $xy$.

*If an algebra with no properly nilpotent element possesses an invariant sub-algebra, it is reducible.*

Let $m$ be the modulus of the algebra $A$, and $e$ the modulus of the invariant sub-algebra $B$. Then $B = eAe$. The sets $(m-e)Ae$ and $eA(m-e)$ are zero, for they belong to $B$ for which $e$ is a modulus, and $e(m-e) = (m-e)e = 0$.

Hence $$A = (m-e)A(m-e)+eAe,$$

and the algebra is reducible.

*If $x$ is an element of an irreducible algebra $A$ with no properly nilpotent element, then $AxA = A$.*

Otherwise $AxA$ would be an invariant sub-algebra of $A$.

We are now in a position to prove the two essential theorems of this chapter.

I. *An irreducible algebra over the complex numbers possessing no properly nilpotent element is a simple matrix algebra.*

It possesses a modulus $m$ which may be expressed as the sum of irreducible idempotents

$$m = e_{11}+e_{22}+e_{33}+\ldots+e_{rr},$$

where $$e_{ii}e_{jj} = 0 \quad (i \neq j).$$

The sub-algebra $e_{ii}Ae_{ii}$ is of order 1, since its modulus $e_{ii}$ is an irreducible idempotent. Denote by $A_{ij}$ the linear set $e_{ii}Ae_{jj}$.

The set $A_{ij}$ cannot be zero, or the set $Ae_{ii}A$ would not contain the element $e_{jj}$, and since the algebra is irreducible and contains no properly nilpotent element, $Ae_{ii}A = A$.

Let $e_{1j}$ be any element of the set $A_{1j}$. Thus there is an element $e_{j1}$ of the set $A_{j1}$ such that $e_{1j}e_{j1} = e_{11}$. $e_{j1}e_{1j}$ is clearly an idempotent of $A_{jj}$, and

$$e_{j1}e_{1j} = e_{jj}.$$

Let $e_{ij} = e_{i1}e_{1j}$. Then the set $A_{ij}$ is of order 1 only, for if not, let $x_{ij}$ be an element of $A_{ij}$ which is not a scalar multiple of $e_{ij}$. Since $x_{ij}e_{ji}$ belongs to $A_{ii}$ we have

$$x_{ij}e_{ji} = \lambda e_{ii},$$

with $\lambda$ a scalar, and $$(x_{ij}-\lambda e_{ij})e_{ji} = 0,$$

and multiplying on the right by $e_{ij}$

$$(x_{ij}-\lambda e_{ij})e_{jj} = 0.$$

But since $x_{ij}-\lambda e_{ij}$ belongs to $e_{ii}Ae_{jj}$, we have $(x_{ij}-\lambda e_{ij})e_{jj} = x_{ij}-\lambda e_{ij}$. Hence $x_{ij} = \lambda e_{ij}$, and we arrive at a contradiction.

Hence the set $A_{ij}$ is of order 1, and the $r^2$ elements $e_{11}, e_{12},..., e_{1r}, e_{21},..., e_{rr}$ form a basis for the algebra, and satisfy

$$\begin{aligned} e_{ij}\,e_{jk} &= e_{ik}, \\ e_{ij}\,e_{kl} &= 0 \quad (j \neq k). \end{aligned}$$

The algebra is a simple matrix algebra.

II. *Every algebra over the complex numbers, with no properly nilpotent element, is expressible as the direct sum of simple matrix algebras.*

## 2.9. The trace of an element

The characteristic equation of an element $x = \sum \xi_r e_r$, i.e. the characteristic equation of the matrix $[\sum \xi_r \gamma_{srt}]$, is independent of the chosen basis of the algebra. Hence the spur of this matrix, $\sum\limits_{ri} \xi_r \gamma_{iri}$ is independent of the basis.

This spur $\sum\limits_{ri} \xi_r \gamma_{iri}$ is called the *trace* of the element $x = \sum \xi_r e_r$ in the algebra. The traces of elements have the following properties.

I. *The trace of a sum is the sum of the traces.* For the trace is linear in the coefficients $\xi_r$.

II. *If an algebra $A$ is the direct sum of algebras $B_1$ and $B_2$, and $x = x_1+x_2$, where $x_1$ and $x_2$ belong to $B_1$ and $B_2$ respectively, then the trace of $x$ in $A$ is the sum of the traces of $x_1$ in $B_1$ and $x_2$ in $B_2$.*

This is seen to be obvious if the basis of $A$ is chosen so that $e_1,..., e_r$ belong to $B_1$, and $e_{r+1},..., e_n$ belong to $B_2$.

III. *If $A$ has a modulus, the trace of the modulus is equal to the order of the algebra.* For, taking $e_1$ as the modulus, $\gamma_{i1i} = 1$.

IV. *The trace of a nilpotent element is zero.* For the matrix $[\sum \xi_r \gamma_{srt}]$ is nilpotent, and its spur is therefore zero.

From the properties of the trace we deduce immediately the following.

V. *In a simple matrix algebra the spur of the matrix corresponding to a given element $x$ is independent of the particular mode of representation as a simple matrix algebra.*

For clearly the trace of a matrix of order $n^2$, considered as an element of a simple matrix algebra, is $n$ times its spur. Since the trace is invariant for a change of basis of the algebra, so also is the spur.

The invariant properties of the trace constitute a powerful weapon, as will be seen in Chapter IV.

# III

# GROUPS

## 3.1. Definition of a group

A *group* is a system composed of a set of elements $a$, $b$, $c$,..., and a rule for combining any two of them to obtain their 'product', such that (1) every product of two elements and the square of each element are elements of the set, (2) the associative law holds, i.e. $a(bc) = (ab)c$, (3) the set contains an identity element $I$ such that $Ia = aI = a$ for every element $a$ of the set, and (4) each element $a$ of the set has an inverse $a^{-1}$ belonging to the set such that

$$aa^{-1} = a^{-1}a = I.$$[†]

If the elements are finite in number, the group is said to be *finite*, and the number of elements is called the *order* of the group. For the present we consider exclusively finite groups.

The operation of forming the product, or multiplication, is not necessarily commutative, i.e. $ab$ and $ba$ may be distinct elements. A group in which all multiplication is commutative is called an *Abelian group*.

If $a$ is an element of a finite group, then $a$, $a^2$, $a^3$,..., $a^n$,... cannot all be distinct. Let $a^r = a^s$ $(r < s)$. Multiplying by $a^{-r}$ we obtain

$$a^{s-r} = I.$$

If $p$ is the least index for which $a^p = I$, $p$ is called the *order* of the element $a$.

A group which consists of a single element $a$ and its powers

$$a,\ a^2,\ a^3,\ \ldots,\ a^p = I$$

is called a *cyclic* group. A cyclic group is clearly Abelian.

Two groups $G$ and $G'$ are said to be *simply isomorphic* if to each element $a$, $b$,... of $G$ there corresponds a unique element $a'$, $b'$,... of $G'$, and conversely, such that $a'b'$ corresponds to $ab$. Two simply isomorphic groups are said to be *equivalent*.

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[†] It is sufficient to assume in (3) that $Ia = a$, and in (4) that $a^{-1}a = I$. It follows that $aa^{-1} = Iaa^{-1} = (a^{-1})^{-1}a^{-1}aa^{-1} = (a^{-1})^{-1}a^{-1} = I$, and $aI = aa^{-1}a = Ia = a$.

The alternative pair of assumptions $aI = a$, $a^{-1}a = I$, however, is not sufficient. They hold for the set of 2-rowed matrices of the form $\begin{bmatrix} a, & 0 \\ b, & 0 \end{bmatrix}$, where $I = \begin{bmatrix} 1, & 0 \\ 0, & 0 \end{bmatrix}$, and these do not form a group, elements having, in general, no right-hand inverse.

If the elements of a group are just symbols which have no interpretation except as elements of the group, the group is called an *abstract group*. The elements may, however, be quantities or operations of a special kind such as matrices, rotations, or permutations. Such a group is called a *special group*, and is said to form a *representation* of the simply isomorphic abstract group.

There is no way of distinguishing two simply isomorphic abstract groups, and they are said to be the same group. Two simply isomorphic special groups may, of course, be distinguished by their other properties. They are said to *possess the same abstract group*.

## 3.2. Subgroups

If in a group of order $h$ there is a set of $g$ $(< h)$ elements which themselves form a group, the set is called a *subgroup*.

Every group which is not cyclic contains a cyclic subgroup, for if $a$ is any element of order $p$, the elements

$$a,\ a^2,\ a^3,\ \ldots,\ a^p = I$$

form a cyclic subgroup. A cyclic group of which the order is not prime also possesses a cyclic subgroup, for if the group consists of the elements

$$a,\ a^2,\ a^3,\ \ldots,\ a^p = I,$$

and $q$ divides $p$, then the elements

$$a^q,\ a^{2q},\ \ldots,\ a^p = I$$

form a subgroup.

Let $G$ be a subgroup of order $g$ of a group $H$ of order $h$, and let $G$ consist of the elements

$$S_1,\ S_2,\ \ldots,\ S_g.$$

Let $T_2$ be any element of $H$ that does not belong to $G$, and denote by $G_2$ the set of elements

$$T_2S_1,\ T_2S_2,\ \ldots,\ T_2S_g.$$

No element of $G_2$ is an element of $G$, or we should have

$$T_2 = (T_2S_r)S_r^{-1}$$

also belonging to $G$.

Let $T_3$ be any element of $H$ which belongs neither to $G$ nor to $G_2$, and denote by $G_3$ the set of elements

$$T_3S_1,\ T_3S_2,\ \ldots,\ T_3S_g.$$

Then no element of $G_3$ belongs either to $G$ or $G_2$, for if, e.g.,

$$T_2S_r = T_3S_p,$$

then
$$T_3 = T_2 S_r S_p^{-1}$$
would belong to $G_2$.

Continuing thus until the elements of $H$ are exhausted, $H$ is divided into $\nu$ distinct subsets $G, G_2, G_3, \ldots, G_\nu$, each of $g$ elements. Thus
$$h = g\nu.$$

*The order of a subgroup is an exact submultiple of the order of the group.*

Two corollaries follow without difficulty.

*The order of every element of a group is an exact submultiple of the order of the group.*

For the element defines a cyclic subgroup of the same order.

*A group of composite order has a subgroup. A group of prime order is cyclic and has no subgroup.*

If $T_1, T_2, \ldots, T_p$ are any elements of a group $H$, then the set of elements formed by multiplying them together in any way forms a group, which is either $H$ or a subgroup of $H$. It is the subgroup of $H$ of least order which contains the given elements. This is called the *group generated by the elements* $T_1, \ldots, T_p$.

The cyclic subgroup $a, a^2, \ldots, a^p = I$ is the group generated by the element $a$.

### 3.3. Examples of groups

Let $ABC$ be an equilateral triangle. Consider the rotations which bring the triangle into coincidence with its original position.

If the rotations are allowed only in the plane of the triangle, these are three in number. There is the rotation $I$ which leaves the triangle unchanged. There is a rotation $S$ which brings the vertex $A$ to the position of $B$, $B$ to the position of $C$, $C$ to the position of $A$; and a rotation $S^2$ which brings $A$ to $C$, $C$ to $B$, and $B$ to $A$.

We define thus a cyclic group of order 3 with elements
$$S,\ S^2,\ S^3 = I.$$

If, however, rotations are allowed in planes perpendicular to the plane of the triangle, so as to interchange the two faces of the triangle which may be regarded as a flat figure in three dimensions, three other rotations are possible. There is a rotation $T$ which interchanges the vertices $B$ and $C$, leaving $A$ unaltered; the rotation $ST = TS^2$ interchanges $A$ and $C$, leaving $B$ unaltered; and the rotation $S^2T = TS$ interchanges $A$ and $B$. We have thus a group of order 6.

The definition of the *product* of two rotations is here taken to be the result of their consecutive operation. A convention as to order is needed, and we assume that the rotation corresponding to a product $ST$ is obtained by taking a rotation $T$ followed by a rotation $S$.

The rotations in the plane, of a regular $n$-sided polygon into itself, form a cyclic group of order $n$. The group of rotations, which includes also rotations in perpendicular planes, is a group of order $2n$, called the *dihedral group of order* $2n$.

Each of the regular solids has its group of rotations.

A set of transformations in the complex variable may form a group. Consider the six fractional transformations

$$w = z, \quad w = -1/(z+1), \quad w = -(z+1)/z,$$
$$w = 1/z, \quad w = -z-1, \quad w = -z/(z+1).$$

Define the product of two transformations as that transformation obtained by their consecutive operation, with the same convention as to order as for rotations. Then these six transformations form a group which is easily shown to be simply isomorphic with the dihedral group of order 6.

The set of six matrices

$$\begin{bmatrix} 1, & 0 \\ 0, & 1 \end{bmatrix}, \quad \begin{bmatrix} 0, & 1 \\ -1, & -1 \end{bmatrix}, \quad \begin{bmatrix} -1, & -1 \\ 1, & 0 \end{bmatrix},$$
$$\begin{bmatrix} 0, & 1 \\ 1, & 0 \end{bmatrix}, \quad \begin{bmatrix} -1, & -1 \\ 0, & 1 \end{bmatrix}, \quad \begin{bmatrix} 1, & 0 \\ -1, & -1 \end{bmatrix},$$

with the usual rule for multiplication, form a group simply isomorphic with the above group of bilinear transformations and the dihedral group of order 6.

Lastly, consider the set of rearrangements of three symbols $\alpha$, $\beta$, $\gamma$. Denote by $I$ the operation of leaving them unchanged; by $S_2$ the operation of replacing $\alpha$ by $\beta$, $\beta$ by $\gamma$, and $\gamma$ by $\alpha$; by $S_3$ the operation of replacing $\alpha$ by $\gamma$, $\gamma$ by $\beta$, and $\beta$ by $\alpha$; by $S_4$ the interchange of $\beta$ and $\gamma$; and by $S_5$ and $S_6$ respectively the interchanges of $\gamma$ and $\alpha$, $\alpha$ and $\beta$. Then these six operations form yet another group simply isomorphic with the dihedral group of order 6.

## 3.4. Permutation groups

A rearrangement of the order of $n$ symbols $(\alpha_1, \alpha_2, \ldots, \alpha_n)$ is called a *permutation*. In a permutation $S$ let $\alpha_{a_1}$ take the place of $\alpha_{a_2}$, $\alpha_{a_2}$ the place of $\alpha_{a_3}$, and so on, until finally $\alpha_{a_p}$ takes the place of $\alpha_{a_1}$. If $\alpha_{b_1}$ is any other symbol whose position is changed, let $\alpha_{b_1}$ take the position of $\alpha_{b_2}$, $\alpha_{b_2}$ that of $\alpha_{b_3}$, and so on, until $\alpha_{b_q}$ takes the position of $\alpha_{b_1}$. Continue thus until every changed symbol has been taken. In this way the permutation $S$ may be specified by the notation

$$(\alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_p})(\alpha_{b_1}\alpha_{b_2}\ldots\alpha_{b_q})(\alpha_{c_1}\ldots)\ldots.$$

The unaltered symbols may or may not be mentioned at the end, according as the terminology is or is not required to convey the totality of symbols on which the permutation operates. The permutation by which $\{a, b, c, d, e, f\}$ is replaced by $\{c, d, e, b, a, f\}$ would thus be represented by

$$(aec)(bd)f \quad \text{or} \quad (aec)(bd).$$

The set of replacements denoted by $(\alpha_{a_1}\alpha_{a_2}\ldots\alpha_{a_p})$ is called a *cycle of order p*. The sum of the orders of the cycles, including the cycles of order 1, i.e. the unchanged symbols, is equal to the number of symbols, $n$.

If $S$ and $T$ are two permutations, the product $ST$ is defined to be the permutation obtained by operating first with the permutation $T$ and then with $S$, so that, e.g.,

$$(ab)(abc) = (ac),$$

$$(abc)(ab) = (bc).$$

The rule for finding the product of two permutations expressed in this terminology is readily explained from the first of these examples. In the first bracket $a$ is followed by $b$, in the second bracket $b$ is followed by $c$. Hence in the product $a$ is followed by $c$. Similarly, since the brackets are interpreted cyclically, so that the first symbol may be regarded as following the last, in $(ab)$ $b$ is followed by $a$, in $(abc)$ $a$ is followed by $b$. Hence, in the product, $b$ is unchanged. Similarly, $c$ is unchanged in $(ab)$, and leads to $a$ in $(abc)$.

A set of permutations on $n$ symbols, which forms a group, is called a *permutation group of degree n*.

The set of all possible permutations on $n$ symbols forms a group of order $n!$, called the *symmetric group of order n!*, or *the symmetric group of degree n*.

Every permutation group of degree $n$ is either the symmetric group of order $n!$ or a subgroup of this group.

## 3.5. The alternating group

A cycle of order 2 is called an *interchange*. Any permutation may be expressed as a product of interchanges, for each cycle $(\alpha_{a_1}\alpha_{a_2}\dots\alpha_{a_p})$ may be expressed as the product

$$(\alpha_{a_1}\alpha_{a_2})(\alpha_{a_1}\alpha_{a_3})(\alpha_{a_1}\alpha_{a_4})\dots(\alpha_{a_1}\alpha_{a_p}).$$

Further, if an initial order $\alpha_1, \alpha_2,\dots, \alpha_n$ is assigned to the symbols, every interchange, and consequently every permutation, may be expressed as a product of interchanges on consecutive symbols, e.g.

$$(\alpha_3\alpha_6) = (\alpha_3\alpha_4)(\alpha_4\alpha_5)(\alpha_5\alpha_6)(\alpha_4\alpha_5)(\alpha_3\alpha_4).$$

Now let any permutation $S$ change $\{\alpha_1, \alpha_2,\dots, \alpha_n\}$ into $\{\alpha_{e_1}, \alpha_{e_2},\dots, \alpha_{e_n}\}$, and denote by $\phi(S)$ the product

$$\phi(S) = \prod (e_r - e_s) \quad (s < r).$$

If $T$ corresponds to an interchange on two consecutive symbols, we have

$$\phi(TS) = -\phi(S),$$

one term of the product being changed in sign.

$S$ is called a *positive* or a *negative* permutation according as $\phi(S)$ is positive or negative. Clearly $\phi(I)$ is positive.

If a positive permutation is expressed as a product of interchanges on consecutive symbols, these must be even in number, since the sign of $\phi$ is changed by each such interchange. Similarly, a negative permutation must be the product of an odd number of interchanges. The following result is therefore apparent.

*The product of two positive or of two negative permutations is positive; the product of a positive and a negative permutation is negative.*

It is easily verified that if the number of cycles of even order is even, the permutation is positive; if it is odd, the permutation is negative.

The positive permutations clearly form a group which is called the *alternating group*. The order of this group is one-half of the order of the corresponding symmetric group, namely $\frac{1}{2}(n!)$, for if $T$ is any fixed negative permutation, for every positive permutation $S$ there is a negative permutation $TS$, and conversely.

The names *symmetric* and *alternating* are also applied to the abstract groups simply isomorphic with the corresponding permutation groups, and, generally, to any simply isomorphic group. Thus we say that the dihedral group of order 6 *is* the symmetric group of that order. The corresponding alternating group is the cyclic group of order 3.

## 3.6. Classes of conjugate elements

Let $S_1$ and $T$ be any two elements of a group. The element $S_2 = T^{-1}S_1 T$ is called the *transform* of $S_1$ by the element $T$. Clearly $S_1$ is the transform of $S_2$ by $T^{-1}$. Two elements which may be transformed into one another are called *conjugate elements.*

Two elements which are transforms of the same element are transforms of one another, for if

$$S_2 = T^{-1}S_1 T, \qquad S_3 = U^{-1}S_1 U,$$

then

$$S_3 = U^{-1}TS_2 T^{-1}U = (T^{-1}U)^{-1}S_2 T^{-1}U.$$

The complete set of elements $S_1, S_2,..., S_r$ which are conjugate to a given element $S_1$, is called a *class of conjugate elements,* or simply a *class* of the group. The number $r$ of such elements is the *order* of the class. In every group the identity element $I$ forms a class of order 1.

Let $H$ be a group of order $h$, and $S_1$ an element of a class $\rho$ of order $h_\rho$. Suppose there are $g$ elements of $H$, $T_1, T_2,..., T_g$ which commute with $S_1$, i.e. such that

$$S_1 T_i = T_i S_1, \qquad T_i^{-1}S_1 T_i = S_1.$$

Let $S_2$ be any other element of the class $\rho$. There is an element $U$ which transforms $S_1$ into $S_2$. Then the $g$ elements $T_i U$ each transform $S_1$ into $S_2$, for

$$(T_i U)^{-1}S_1 T_i U = U^{-1}T_i^{-1}S_1 T_i U = U^{-1}S_1 U = S_2.$$

Again, if $V$ transforms $S_1$ into $S_2$,

$$(VU^{-1})^{-1}S_1 VU^{-1} = UV^{-1}S_1 VU^{-1} = US_2 U^{-1} = S_1,$$

and $VU^{-1}$ commutes with $S_1$, and is one of the elements $T_i$.

Hence there are exactly $g$ elements of $H$ which transform $S_1$ into $S_2$, and in general exactly $g$ elements which transform $S_1$ into any other element of the class $\rho$. Since there are $h$ elements of $H$ by which $S_1$ may be transformed, it follows that

$$h = gh_\rho,$$

and we have

I. *The order of any class is an exact submultiple of the order of the group.*

If $S$ is a fixed element of a class $\rho$, and $T$ runs through the $h$ elements of the group, the $h$ elements $T^{-1}ST$ consist of the class $\rho$ repeated $h/h_\rho$ times.

In the case of the symmetric group of permutations on $n$ symbols, two operations which have the same number of cycles of the same orders are conjugate, for if

$$S_1 = (\alpha_{11}\,\alpha_{12}\,\ldots\,\alpha_{1p})(\alpha_{21}\,\ldots\,\alpha_{2q})\ldots,$$

and

$$S_2 = (\beta_{11}\,\beta_{12}\,\ldots\,\beta_{1p})(\beta_{21}\,\ldots\,\beta_{2q})\ldots,$$

then the operation which replaces $\{\alpha_{11}\,\alpha_{12}\,\ldots\,\alpha_{1p}\,\alpha_{21}\,\ldots\,\alpha_{2q}\,\ldots\}$ by $\{\beta_{11}\,\beta_{12}\,\ldots\,\beta_{1p}\,\beta_{21}\,\ldots\}$ transforms $S_2$ into $S_1$. Clearly any transform of a permutation has the same number of cycles of the same orders, and the classes of the symmetric group are defined by the number and orders of the cycles. Each of these sets of cycles may in turn be associated with a *partition*† of $n$, i.e. with an expression of $n$ as a sum of positive integers.

II. *The number of classes of the symmetric group of order $n!$ is equal to the number of partitions of $n$.*

As an example, the symmetric group of order 6 has three classes. The first consists of the identity, which has three cycles of order 1. The corresponding partition is $3 = 1+1+1$, which is written $(1, 1, 1)$ or shortly $(1^3)$. The second class contains the three substitutions

$$(\alpha_1\,\alpha_2), \qquad (\alpha_1\,\alpha_3), \qquad (\alpha_2\,\alpha_3).$$

There is one cycle of order 2 and one cycle of order 1, and the partition is $(2\ 1)$. The third class comprises the substitutions

$$(\alpha_1\,\alpha_2\,\alpha_3), \qquad (\alpha_1\,\alpha_3\,\alpha_2),$$

corresponding to the partition $(3)$.

It may be verified that there are five classes of the symmetric group of order $4!$, corresponding to the partitions $(1^4)$, $(1^2\ 2)$, $(1\ 3)$, $(4)$, and $(2^2)$, and of orders 1, 6, 8, 6, and 3 respectively.

The order of the class $(1^{a_1}\,2^{a_2}\,3^{a_3}\,\ldots)$ of the symmetric group of order $n!$ may be found as follows. To any permutation of the $n$ symbols there corresponds a substitution of this class, obtained by converting the first $a_1$ symbols into cycles of order 1, the following $2a_2$ symbols, in pairs, into cycles of order 2, and so on. Two distinct permutations, however, may lead to the same substitution, for a cycle of order $r$ may be written commencing with any symbol of the cycle, and, further, the cycles of equal order may be permuted amongst themselves. There are thus $r^{a_r} a_r!$ ways of permuting the symbols in the cycles of order $r$ without changing the corresponding substitution.

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† See Chapter V.

Altogether $1^{a_1} a_1!\ 2^{a_2} a_2!\ 3^{a_3} a_3! \dots$ permutations of the $n$ symbols each give the same substitution of the class. Since there are $n!$ permutations of the $n$ symbols, it follows that the order of the class is

$$h_\rho = n!/1^{a_1} a_1!\ 2^{a_2} a_2!\ 3^{a_3} a_3! \dots. \tag{3.6; 1}$$

The elements of a class $\rho$ of a group $H$ do not necessarily belong to one class in a subgroup $G$. For it may be that none of the elements of $H$ which transform $S_1$ into $S_2$ belong to $G$. In this case $S_1$ and $S_2$ belong to different classes in the subgroup. Thus one class of a group may separate into any number of classes in a subgroup. Or, again, it may happen that there are no members of a class of the group which belong to the subgroup. On the other hand, one class of the subgroup must correspond to one and only one class of the group. For the elements of the subgroup $G$ all belong to the group $H$, and if an element of $G$ transforms $S_1$ into $S_2$, this element is also an element in $H$, and $S_1$ and $S_2$ are conjugate in $H$.

If a class consists of one element only, this element is called a *self-conjugate element.*

A self-conjugate element commutes with every element of the group, and, conversely, an element which commutes with every element of the group is self-conjugate. Every element of an Abelian group is self-conjugate.

### 3.7. Conjugate and self-conjugate subgroups

Let a group $H$ have a subgroup $G$ consisting of the elements

$$S_1, S_2, \dots, S_g.$$

Let $T$ be any element of $H$. Then the elements

$$T^{-1}S_1T,\ T^{-1}S_2T,\ \dots,\ T^{-1}S_gT$$

clearly form a group, for

$$(T^{-1}S_iT)(T^{-1}S_jT) = T^{-1}S_iS_jT.$$

This group may coincide with $G$. Otherwise it is called a *transform of the group* $G$, and may be denoted by $T^{-1}GT$. $G$ and $T^{-1}GT$ are called *conjugate subgroups* of $H$.

The complete set of subgroups conjugate with a given subgroup is called a *class of conjugate subgroups.*

If a class of conjugate subgroups consists of the one group $G$ only, then $G$ is called a *self-conjugate subgroup* of $H$. If $S$ is any element of $G$, and $T$ any element of $H$, then $T^{-1}ST$ must belong to $G$ if $G$ is self-conjugate in $H$.

Each self-conjugate subgroup $G$ of order $g$ of a group $H$ of order $h$, defines another group of order $\nu = h/g$ called the *quotient group.*

Let $S_1, S_2,..., S_g$ be the elements of $G$. Then, as shown in § 3.2 we can find $\nu$ elements of $H$, $T_1, T_2,..., T_\nu$ such that the sets $T_1 G$, $T_2 G,..., T_\nu G$ include all the elements of $H$.

Suppose that $T_i T_j$ belongs to $T_k G$, so that $T_k^{-1} T_i T_j$ belongs to $G$. Let $U_i$ and $U_j$ be any other pair of elements from $T_i G$ and $T_j G$ respectively, so that $T_i^{-1}U_i$ and $T_j^{-1}U_j$ belong to $G$, and hence also, since $G$ is self-conjugate, $U_j^{-1}(T_i^{-1}U_i)U_j$. Then

$$T_j^{-1}U_j . U_j^{-1}(T_i^{-1}U_i)U_j = (T_i T_j)^{-1}U_i U_j$$

belongs to $G$, and $U_i U_j$ and $T_i T_j$ belong to the same set $T_k G$.

Take $\nu$ elements $\Gamma_1, \Gamma_2,..., \Gamma_\nu$ with multiplication table

$$\Gamma_i \Gamma_j = \Gamma_k$$

whenever $T_i T_j$ belongs to $T_k G$. These elements form the *quotient group* denoted by $H/G$.

The quotient group $H/G$ is multiply isomorphic with the group $H$, the element $\Gamma_i$ corresponding to the $g$ elements of the set $T_i G$. If $U_i$ and $U_j$ are any pair of elements of $H$ corresponding to $\Gamma_i$ and $\Gamma_j$ respectively, then $U_i U_j$ corresponds to $\Gamma_i \Gamma_j$.

## 3.8. The representations of an abstract group as a permutation group

Let $G$ be a subgroup of order $g$ of a group $H$ of order $h$. Let $T_1, T_2,..., T_\nu$ be $\nu = h/g$ elements such that

$$T_1 G,\ T_2 G,\ ...,\ T_\nu G$$

include the $h$ elements of $H$.

Let $U$ be any element of $H$. If $UT_i$ belongs to $T_j G$, then the set $UT_i G$ is identical with the set $T_j G$, for if $S$ is any element of $G$,

$T_j^{-1}UT_i$ belongs to $G$, and hence also $T_j^{-1}UT_iS$, so that $UT_iS$ belongs to $T_jG$.

Hence the sets

$$UT_1G,\ UT_2G,\ \ldots,\ UT_\nu G$$

are the same as the sets $T_1G$, $T_2G, \ldots$, $T_\nu G$, except that the order may be different. We thus obtain a permutation $P$ of the sets, corresponding to the element $U$. The permutation $Q$ of the sets, corresponding to another element $V$ of $H$, replaces $UT_iG$ by $VUT_iG$. Hence the product $QP$ of the permutations replaces $T_iG$ by $VUT_iG$ and corresponds to the product $VU$.

The permutations of the sets are isomorphic with the elements of the group $H$, and form a *representation of the group $H$ as a permutation group, corresponding to the subgroup $G$.*

This representation will be *simply* isomorphic if each permutation corresponds to a single element $U$ of $H$. If it is not simply isomorphic, there is an element $U$ of $H$ which does not permute the sets. Thus $UT_iG$ will belong to $T_iG$ for all $i$, so that $T_i^{-1}UT_i$ belongs to $G$. Since $T_i$ may be replaced by any element of the set $T_iG$, it follows that the transform of $U$ by any element of $H$ must belong to $G$. The elements of the class of $H$ to which $U$ belongs must generate a self-conjugate subgroup of $H$ which is either $G$ or a subgroup of $G$. This necessary condition for multiple isomorphy is also sufficient.

*The necessary and sufficient condition that the isomorphy of the permutation representation of a group $H$ corresponding to a subgroup $G$ should be simple, is that neither $G$ nor any subgroup of $G$ should be self-conjugate in $H$.*

The set which consists of the identical element only may be regarded as the limiting case of a subgroup of $H$. There is a corresponding permutation representation of degree $h$, in which the $h$ elements of the group $H$ themselves are permuted. The isomorphy is necessarily simple. This representation is called the *regular permutation representation of the group.*

### Transitive permutation groups

A *transitive* permutation group is one which contains a permutation replacing any given symbol by any other; in an *r-ply transitive* group any set of $r$ symbols may be replaced simultaneously by any other such set. In an *intransitive* group the symbols are divided into *transitive sets*, the symbols of each set being permuted amongst themselves.

# IV

# THE FROBENIUS ALGEBRA

## 4.1. Groups and algebras

A SIMILARITY between the properties of groups and algebras, especially the property of non-commutative multiplication, suggests a connexion between the two theories. In fact, if we take an abstract group, and further define operations of addition and scalar multiplication, we obtain a special type of algebra which is called after Frobenius, the founder of the theory, a *Frobenius algebra* (**1**).†

Let $S_1, S_2, \ldots, S_h$ be the operations of a group $H$. Then the algebra with basis $e_1, e_2, \ldots, e_h$ and multiplication table

$$e_i e_j = e_k \quad \text{whenever } S_i S_j = S_k,$$

so that

$$\gamma_{ijk} = 1 \quad \text{if } S_i S_j = S_k,$$
$$\gamma_{ijk} = 0 \quad \text{if } S_i S_j \neq S_k,$$

is called the *group algebra*, or the *Frobenius algebra* of the group.

There is a logical distinction between the elements of the group and the basal elements of the Frobenius algebra. Henceforward, however, we shall use the same symbol $S_i$ for both purposes. No logical confusion arises, as the basal elements do in any case form a representation of the group, and the exposition is thus made simpler.

It is necessary to distinguish between an *element of the algebra*, which is any element of the form $\sum \xi_i S_i$, and a *group element*, which must be one of the basal elements of the algebra $S_i$.

We replace $S_h$ by $S_0$, and use this symbol to denote the identical element of the group. From the multiplication table we obtain

$$\gamma_{j0j} = 1, \qquad \gamma_{jij} = 0 \quad (i \neq 0).$$

We thus obtain the following result.

*The trace of $S_0$ is $h$, and the trace of every other group element is zero.*

Thus the trace of $X = \sum \xi_i S_i$ is $h\xi_0$.

THEOREM. *The Frobenius algebra of a group is expressible as a direct sum of simple matrix algebras.*

This is the important theorem of this section. The proof follows immediately from the following lemma.

LEMMA. *The Frobenius algebra of a group contains no properly nilpotent element.*

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† Bold-face numbers refer to notes at end of book.

Let $X = \sum \xi_i S_i$ be any element of the algebra, and let

$$\bar{X} = \sum \bar{\xi}_i S_i^{-1},$$

where $\bar{\xi}_i$ is the conjugate complex number of $\xi_i$. Then the trace of $X\bar{X}$, namely $h \sum \xi_i \bar{\xi}_i$, is clearly a positive number. It follows that $X\bar{X}$ cannot be nilpotent, and $X$ cannot be properly nilpotent. This proves the lemma, and, consequently, the theorem.

**Elements which commute with every element of the algebra**

Let the group $H$ contain $p$ classes, $C_0, C_1, \ldots, C_{p-1}$, of which $C_0$ is the class containing the identity. We shall use the word *class* and the symbol $C_\rho$ also to denote the sum of the elements of the class.

Let $S_i$ be any group element of the class $C_\rho$ of order $h_\rho$. Then, from § 3.6, if $S_j$ runs through the $h$ elements of $H$, the $h$ elements $S_j^{-1}S_iS_j$ consist of the class $C_\rho$ repeated $h/h_\rho$ times. Hence

$$\sum_j S_j^{-1}S_iS_j = hC_\rho/h_\rho.$$

Thus, if $X$ is any element $\sum \xi_i S_i$ of the algebra, $\sum_j S_j^{-1}XS_j$ is a linear function of the classes.

If an element $X$ of the algebra commutes with every element of the algebra, then

$$S_j^{-1}XS_j = X$$

and

$$\frac{1}{h}\sum_j S_j^{-1}XS_j = X.$$

It follows that $X$ is a linear function of the classes. Conversely, a linear function of the classes commutes with every element of the algebra.

*An element of a Frobenius algebra which commutes with every other element of the algebra is a linear function of the classes, and, conversely, a linear function of the classes commutes with every element of the algebra.*

Such elements form a linear set of order $p$.

Let the Frobenius algebra be equivalent to the direct sum of $q$ simple matrix sub-algebras $\Gamma_1, \Gamma_2, \ldots, \Gamma_q$, and let the modulus of the sub-algebra $\Gamma_i$ be $\epsilon_i$.

The only matrices which commute with every matrix of order $n^2$ are the scalar multiples of the unit matrix. Thus, if an element commutes with every element of the Frobenius algebra, its representation in each sub-algebra is a scalar multiple of the modulus, and the element may be expressed in the form $\sum \lambda_i \epsilon_i$, $\lambda_i$ being scalar. The order of the linear set of such elements is $q$, and it follows that $p = q$.

*The number of simple matrix sub-algebras in the Frobenius algebra of a group is equal to the number of classes.*

The classes can clearly be expressed linearly in terms of the moduli of the sub-algebras, and conversely.

$$C_\rho = \sum \psi_\rho^{(i)} \epsilon_i, \tag{4.1; 1}$$

$$\epsilon_i = \sum \phi_\rho^{(i)} C_\rho. \tag{4.1; 2}$$

Denote the class containing the inverses of the elements in the class $C_\rho$ by $C_{\rho'}$. By substituting from (4.1; 1) and (4.1; 2) respectively we have

$$\begin{aligned} \epsilon_i C_\rho &= \sum \psi_\rho^{(j)} \epsilon_i \epsilon_j = \psi_\rho^{(i)} \epsilon_i \\ &= \sum \phi_\sigma^{(i)} C_\sigma C_\rho. \end{aligned} \tag{4.1; 3}$$

Now the trace of $C_\sigma C_\rho$ is $hh_\rho$ if $\sigma = \rho'$, but zero otherwise. Also the trace of $\epsilon_i$ is $f^{(i)2}$, where $f^{(i)}$ is the dègree of the matrix sub-algebra, i.e. the number of rows or columns.

Taking the trace of (4.1; 3), we have

$$\psi_\rho^{(i)} f^{(i)2} = hh_\rho \phi_{\rho'}^{(i)}. \tag{4.1; 4}$$

Again, taking the trace of (4.1; 2) and remembering that the trace of $C_0$ is $h$, but the trace of every other class is zero, we have

$$f^{(i)2} = h\phi_0^{(i)}. \tag{4.1; 5}$$

Substituting in (4.1; 4), we obtain

$$\psi_\rho^{(i)} = h_\rho \phi_{\rho'}^{(i)} / \phi_0^{(i)},$$

whence (4.1; 1) may be rewritten

$$C_\rho = \sum h_\rho \phi_{\rho'}^{(i)} \epsilon_i / \phi_0^{(i)} \tag{4.1; 6}$$

## 4.2. The group characters

The Frobenius algebra is equivalent to the direct sum of $p$ simple matrix sub-algebras. Any group element $S_i$ may thus be expressed as a sum of elements in each of the sub-algebras. These are called the *representations* of $S_i$ in the various sub-algebras. There are an infinity of ways of expressing each sub-algebra as a simple matrix algebra, but the spur of the matrix is independent of the mode of representation (§ 2.9, Theorem V).

DEFINITION. *The spur of the matrix representation of $S_i$ in the sub-algebra $\Gamma_j$ is called the characteristic of $S_i$, and is written $\chi^{(j)}(S_i)$. The set of characteristics of the h group elements corresponding to the sub-algebra $\Gamma_j$ is called a group character, and is written $\chi^{(j)}$.*

Since the $h_\rho$ elements of a class $C_\rho$ are transforms of one another, the matrix representations in a given sub-algebra have the same spur. Hence the characteristics of the $h_\rho$ elements are equal. This value will be referred to as the *characteristic of the class* $C_\rho$, and may be written $\chi_\rho^{(i)}$. $\chi_0^{(i)} = f^{(i)}$ is called the *degree* of the character.

There are thus $p^2$ distinct numbers which are the characteristics of the $p$ classes corresponding to the $p$ sub-algebras. These numbers may be arranged in a square table which we call the *table of characters*. It will be seen from Chapter IX that almost all the properties of a group may be deduced from this table of characters.

The group characters satisfy very important orthogonal relations, which we proceed to obtain.

The spur of the representation in $\Gamma_i$ of the class $C_\rho$ is evidently $h_\rho \chi_\rho^{(i)}$. Hence the trace of this element, which is the trace of $\epsilon_i C_\rho$, is

$$h_\rho f^{(i)} \chi_\rho^{(i)}.$$

From (4.1; 4) we obtain

$$h_\rho f^{(i)} \chi_\rho^{(i)} = h h_\rho \phi_{\rho'}^{(i)},$$

so that

$$\phi_\rho^{(i)} = f^{(i)} \chi_{\rho'}^{(i)}/h. \tag{4.2; 1}$$

Substituting in equations (4.1; 2), (4.1; 6), we obtain

$$\epsilon_i = \sum f^{(i)} \chi_{\rho'}^{(i)}\, C_\rho/h, \tag{4.2; 2}$$

$$C_\rho = \sum h_\rho \chi_\rho^{(i)}\, \epsilon_i/f^{(i)}. \tag{4.2; 3}$$

In the left-hand side of the equations

$$\epsilon_i^2 = \epsilon_i, \qquad \epsilon_i \epsilon_j = 0$$

substitute from (4.2; 2) and take the trace. Remembering that the trace of $C_\rho C_\sigma$ is $hh_\rho$ if $\sigma = \rho'$, and zero otherwise, we have

$$\sum f^{(i)2} \chi_\rho^{(i)} \chi_{\rho'}^{(i)}\, hh_\rho/h^2 = f^{(i)2},$$

and

$$\sum f^{(i)} f^{(j)} \chi_\rho^{(i)} \chi_{\rho'}^{(j)}\, hh_\rho/h^2 = 0.$$

Thus

$$\left.\begin{aligned} \sum_\rho h_\rho \chi_\rho^{(i)} \chi_{\rho'}^{(i)} &= h, \\ \sum_\rho h_\rho \chi_\rho^{(i)} \chi_{\rho'}^{(j)} &= 0 \quad (i \neq j). \end{aligned}\right\} \tag{4.2; 4}$$

Again, substituting from (4.2; 3) in $C_\rho C_\sigma$ and taking the trace, we have

$$\begin{aligned} \sum h_\rho h_\sigma \chi_\rho^{(i)} \chi_\sigma^{(i)} f^{(i)2}/f^{(i)2} &= hh_\rho \quad \text{if } \sigma = \rho', \\ &= 0 \quad \text{otherwise.} \end{aligned}$$

Hence

$$\left.\begin{aligned} h_\rho \sum_i \chi_\rho^{(i)} \chi_{\rho'}^{(i)} &= h, \\ \sum \chi^{(i)} \chi^{(i)} &= 0 \quad (\sigma \neq \rho') \end{aligned}\right\} \tag{4.2; 5}$$

Equations (4.2; 4) and (4.2; 5) constitute the orthogonal relations referred to above.

EXAMPLE. Consider the symmetric group of permutations on the three symbols $\alpha$, $\beta$, $\gamma$. The six elements of the group include the identity $I$, the three elements

$$(\alpha\beta), \quad (\beta\gamma), \quad (\alpha\gamma),$$

which form the class (2 1), which we will denote by $C_1$; and the two elements

$$(\alpha\beta\gamma), \quad (\alpha\gamma\beta)$$

which form the class (3), which we shall denote by $C_2$.

The multiplication table of the classes, which are, of course, commutative, is as follows:

$$C_0^2 = C_0, \quad C_0 C_1 = C_1, \quad C_0 C_2 = C_2,$$
$$C_1^2 = 3C_0+3C_2, \quad C_1 C_2 = 2C_1, \quad C_2^2 = 2C_0+C_2.$$

Hence, clearly, the moduli of the three simple matrix algebras of the Frobenius algebra are respectively

$$\tfrac{1}{6}(C_0+C_1+C_2),$$
$$\tfrac{1}{6}(C_0-C_1+C_2),$$
$$\tfrac{1}{3}(2C_0-C_2);$$

these being the irreducible idempotents of the algebra of the classes.

The six elements

$$x = \tfrac{1}{6}[I+(\alpha\beta)+(\beta\gamma)+(\gamma\alpha)+(\alpha\beta\gamma)+(\alpha\gamma\beta)],$$
$$y = \tfrac{1}{6}[I-(\alpha\beta)-(\beta\gamma)-(\gamma\alpha)+(\alpha\beta\gamma)+(\alpha\gamma\beta)],$$
$$z_{11} = \tfrac{1}{3}[I+(\alpha\beta)-(\alpha\gamma)-(\alpha\beta\gamma)],$$
$$z_{22} = \tfrac{1}{3}[I+(\alpha\gamma)-(\alpha\beta)-(\alpha\gamma\beta)],$$
$$z_{12} = \tfrac{1}{3}[(\beta\gamma)-(\alpha\gamma)+(\alpha\gamma\beta)-(\alpha\beta\gamma)],$$
$$z_{21} = \tfrac{1}{3}[(\beta\gamma)-(\alpha\beta)+(\alpha\beta\gamma)-(\alpha\gamma\beta)]$$

form a basis for the Frobenius algebra, and satisfy

$$x^2 = x, \quad y^2 = y, \quad xy = yx = 0,$$
$$xz_{ij} = z_{ij}x = yz_{ij} = z_{ij}y = 0,$$
$$z_{ij}z_{jk} = z_{ik}, \quad z_{ij}z_{kp} = 0 \quad (j \neq k;\ i, j, k, p = 1, 2).$$

The Frobenius algebra is thus exhibited as the direct sum of three sub-algebras, two being simply isomorphic with the complex numbers, and the third being a simple matrix algebra of order 4.

Now putting $z = z_{11}+z_{22}$, the modulus of the third sub-algebra, denote $\lambda_{11}z_{11}+\lambda_{12}z_{12}+\lambda_{21}z_{21}+\lambda_{22}z_{22}$ by $\begin{bmatrix}\lambda_{11}, & \lambda_{12}\\ \lambda_{21}, & \lambda_{22}\end{bmatrix}z$. Then we can solve the above equations for the group elements in terms of $x$, $y$, and $z$.

$$I = x+y+\begin{bmatrix}1 & \\ & 1\end{bmatrix}z,$$

$$(\alpha\beta) = x-y+\begin{bmatrix}1, & -1\\ & -1\end{bmatrix}z,$$

$$(\beta\gamma) = x-y+\begin{bmatrix} & 1\\ 1 & \end{bmatrix}z,$$

$$(\gamma\alpha) = x-y+\begin{bmatrix}-1 & \\ -1, & 1\end{bmatrix}z,$$

$$(\alpha\beta\gamma) = x+y+\begin{bmatrix} & -1\\ 1, & -1\end{bmatrix}z,$$

$$(\alpha\gamma\beta) = x+y+\begin{bmatrix}-1, & 1\\ -1 & \end{bmatrix}z.$$

The characters are the spurs of the matrix coefficients of $x$, $y$, and $z$, and may be expressed in table form:

| Class . . | $(1^3)$ | $(2\ 1)$ | $(3)$ |
|---|---|---|---|
| Order . . | 1 | 3 | 2 |
| | 1 | 1 | 1 |
| | 1 | −1 | 1 |
| | 2 | 0 | −1 |

It is easily verified that these characters satisfy the orthogonal relations (4.2; 4) and (4.2; 5).

## 4.3. Matrix representations and group matrices

If to each element $S_i$ of a group there corresponds a matrix $M_i$ such that $M_i M_j = M_k$ whenever $S_i S_j = S_k$, the matrices $M_i$ are said to form a *matrix representation* of the group.

The matrices $M_i$ need not all be distinct. To several elements of the group may correspond identical matrices, so that the representation is not simply, but multiply isomorphic with the group. For example, $M_i$ may be the one-rowed unit matrix for every element of the group. But to each group element $S_i$ there must correspond a unique matrix $M_i$.

The matrix $\sum \xi_i M_i$ which corresponds to the general element $\sum \xi_i S_i$ of the Frobenius algebra is called a *group matrix.* It exhibits in the form of one matrix the complete matrix representation of the group. Each element of the matrix is a linear function of the $\xi_i$'s.

Clearly the sets of linear substitutions to which the matrices $M_i$ correspond form a representation of the abstract group as a group of linear substitutions. The theories of these two types of representation are equivalent.

Since the Frobenius algebra is equivalent to the direct sum of $p$ simple matrix algebras, to each group element there corresponds a matrix in each sub-algebra. The $p$ sub-algebras thus give $p$ matrix representations of the group. We shall show that every *irreducible* matrix representation of the group is *equivalent* to one of these $p$ representations, and every representation is equivalent to a *direct sum* of these representations, each being repeated any number of times, or omitted.

If $X$ is a group matrix of order $n^2$, and $T$ is any fixed matrix of order $n^2$, then $T^{-1}XT$ is also a group matrix, for

$$T^{-1}M_i T \,.\, T^{-1}M_j T = T^{-1}M_i M_j T,$$

and the matrices $T^{-1}M_i T$ are simply isomorphic with the matrices $M_i$.

The group matrices $X$ and $T^{-1}XT$ are said to be *equivalent,* and the corresponding matrix representations are said to be *equivalent.*

If a group matrix $X$ is equivalent to a matrix of the form

$$\left[\begin{array}{c:c} X_1 & Y \\ \hdashline 0 & X_2 \end{array}\right], \tag{4.3;1}$$

where $X_1$ and $X_2$ are square matrices, $X$ is said to be *reducible.*

If, also, it can be arranged so that the rectangular matrix $Y$ in the top right-hand corner is zero as well as the rectangular matrix in the bottom left-hand corner, then $X$ is said to be *completely reducible.*

In this case $X_1$ and $X_2$ are group matrices, and $X$ is said to be equivalent to the *direct sum* of the group matrices $X_1$ and $X_2$.

The group matrices corresponding to the representations obtained in the Frobenius algebra are obviously *irreducible,* for as the corresponding sub-algebra is a *simple* matrix algebra, the $f^2$ elements of the group matrix will be linearly independent. As we shall show that the general group matrix is equivalent to a direct sum of irreducible group matrices corresponding to representations obtained in the Frobenius algebra, it will follow that for finite groups *reducibility implies complete reducibility.*

We assume therefore that in (4.3; 1), $Y = 0$, and by *reducible* we shall mean *completely reducible.*

The matrix representation corresponding to $X$ is also said to be *reducible,* and equivalent to the *direct sum* of the representations corresponding to $X_1$ and $X_2$.

Now let $X = \sum \xi_i M_i$ be any group matrix.

If the determinant $|X|$ is identically zero, let $M_0$ be the matrix corresponding to the identity. Since

$$M_0^2 = M_0,$$

$M_0$ may be transformed into diagonal form. After transformation let

$$M_0 = \operatorname{diag}(1^r, 0^{n-r}).$$

Since $X = M_0 X M_0$, $X$ has now a square matrix of order $r^2$ in the top left-hand corner bordered by $n-r$ rows and columns of zeros. Clearly we may ignore the zeros and consider only the square matrix of order $r^2$. The determinant of this matrix is not identically zero, for the coefficient of $\xi_0^r$ is unity. We assume henceforward that $X$ is a square matrix of non-vanishing determinant, in which the coefficient of $\xi_0$ is the unit matrix.

Let $\epsilon_1, \epsilon_2,\dots, \epsilon_p$ be the moduli of the simple matrix sub-algebras of the Frobenius algebra. Let $m_1, m_2,\dots, m_p$ be the corresponding matrices in the representation, some of which may be identically zero.

Since $\epsilon_i^2 = \epsilon_i$ and $\epsilon_i \epsilon_j = 0$ $(i \neq j)$, we have also

$$m_i^2 = m_i, \qquad m_i m_j = 0 \quad (i \neq j).$$

It follows from § 1.8, Theorem VI, that the matrices $m_1, m_2,\dots, m_p$ may be transformed simultaneously into diagonal form. Let us assume them to be so transformed, and that

$$m_1 = \operatorname{diag}(1^{\alpha_1}, 0^{r-\alpha_1}),$$
$$m_2 = \operatorname{diag}(0^{\alpha_1}, 1^{\alpha_2}, 0^{r-\alpha_1-\alpha_2}),$$

Now if $x$ is any number of the Frobenius algebra,

$$x = \sum \epsilon_i x \epsilon_i.$$

Hence $$X = \sum m_i X m_i.$$
It follows that $X$ is of the form

$$X = \begin{array}{ccc} X_1 & & 0 \\ & & \\ & & \\ & 0 & \end{array}$$

and $X$ is equivalent to the direct sum of the group matrices $X_1$, $X_2,\ldots, X_p$.

We consider now one of these group matrices $X_i$. In this representation the matrix corresponding to $\epsilon_i$ is the unit matrix, and the matrix corresponding to $\epsilon_j$ $(j \neq i)$ is identically zero.

Let the corresponding sub-algebra of the Frobenius algebra be a simple matrix algebra of order $f^2$ with basal units $e_{\alpha\beta}$ $(1 \leqslant \alpha, \beta \leqslant f)$ such that
$$e_{\alpha\beta}\, e_{\beta\gamma} = e_{\alpha\gamma},$$
$$e_{\alpha\beta}\, e_{\gamma\delta} = 0 \quad (\beta \neq \gamma).$$
Let the corresponding matrices in the representation corresponding to the group matrix $X_i$ be $\psi_{\alpha\beta}$.

Now $$\epsilon_i = e_{11}+e_{22}+\ldots+e_{ff}$$
and consequently $$I = \psi_{11}+\psi_{22}+\ldots+\psi_{ff}.$$
Also $$\psi_{jj}^2 = \psi_{jj}, \qquad \psi_{jj}\psi_{kk} = 0 \quad (j \neq k).$$
Hence the matrices $\psi_{11}, \psi_{22},\ldots, \psi_{ff}$ may be transformed simultaneously into diagonal form, and since they may be transformed into one another, they are of equal rank $\nu$ such that
$$f\nu = r.$$
We will assume that after such a transformation
$$\psi_{11} = \mathrm{diag}(1^{\nu}, 0^{r-\nu}),$$
$$\psi_{22} = \mathrm{diag}(0^{\nu}, 1^{\nu}, 0^{r-2\nu}),$$

Since $\psi_{\alpha\beta} = \psi_{\alpha\alpha}\psi_{\alpha\beta}\psi_{\beta\beta}$, it is clear that the matrix $\psi_{\alpha\beta}$ has non-zero terms only in the rows for which $\psi_{\alpha\alpha}$ has unity in the leading diagonal, and in the columns for which $\psi_{\beta\beta}$ has unity in the leading diagonal.

The element $\sum \xi_j S_j$ of the Frobenius algebra may be expressed in the form

$$\sum \xi_j S_j = \sum \zeta_{\alpha\beta} e_{\alpha\beta} + \eta,$$

where $\eta$ belongs to sub-algebras other than $\Gamma_i$. Hence

$$X_i = \sum \zeta_{ij} \psi_{ij},$$

and this may be expressed as a matrix with matrix elements

$$X_i = [\zeta_{st} A_{st}];$$

$A_{st}$ being the $\nu$-rowed square matrix picked out from $\psi_{st}$, from the rows for which $\psi_{ss}$ has unity in the leading diagonal, and the columns for which $\psi_{tt}$ has unity in the leading diagonal.

The matrices $A_{st}$ satisfy

$$A_{\alpha\beta} A_{\beta\gamma} = A_{\alpha\gamma},$$
$$A_{\alpha\alpha} = I_\nu.$$

Now let $$Y = \operatorname{diag}(A_{11}, A_{21}, \ldots, A_{f1}),$$

so that $$Y^{-1} = \operatorname{diag}(A_{11}, A_{12}, \ldots, A_{1f}).$$

Then $$Y^{-1} X_i Y = [\zeta_{st} A_{1s} A_{st} A_{t1}]$$
$$= [\zeta_{st} I_\nu],$$

which by a rearrangement of rows and columns becomes

$$\begin{array}{cccc} [\zeta_{st}] & & 0 & \\ & [\zeta_{st}] & & \\ & & [\zeta_{st}] & \\ & 0 & & \ddots \end{array}$$

and $X_i$ is equivalent to the representation of the simple matrix algebra $\Gamma_i$ repeated $\nu$ times.

*Any group matrix $X$ is equivalent to a direct sum of the irreducible group matrices which correspond to the simple matrix sub-algebras of the Frobenius algebra, each of these group matrices being repeated any number of times, or omitted.*

The spur of $X$ is a linear function of the $\xi$'s, say $\sum \xi_j \phi(S_j)$. The spur of the irreducible group-matrix corresponding to $\Gamma_i$ is $\sum \xi_j \chi^{(i)}(S_j)$.

Hence $$\phi(S_j) = \sum \nu_i \chi^{(i)}(S_j).$$

The coefficient $\nu_i$, being the number of times the representation is repeated, is a positive integer.

*Any* linear function of the characters with positive integral coefficients is called a *compound character*.

*The spurs of the matrices in any matrix representation of the group is a compound character of the group, the coefficient of any simple character being the number of times the corresponding representation is repeated in the equivalent direct sum of irreducible representations.*

Given the compound character $\phi(S_j)$, the coefficients of the simple characters may be found as follows.

Let $$\phi(S_j) = \sum \nu_i \chi^{(i)}(S_j).$$

Then $$\sum \chi^{(i)}(S_j^{-1})\phi(S_j) = \sum \nu_k \chi^{(i)}(S_j^{-1})\chi^{(k)}(S_j) = h\nu_i,$$

so that $$\nu_i = \frac{1}{h}\sum \chi^{(i)}(S_j^{-1})\phi(S_j) = \frac{1}{h}\sum h_\rho \chi_{\rho'}^{(i)} \phi_\rho. \tag{4.3; 2}$$

Given any group $H$ of order $h$, there is a regular permutation representation of degree $h$ (§ 3.8) which may be regarded as a group of linear substitutions, and a corresponding representation by permutation matrices. The corresponding group matrix is called the *regular group matrix*. The spur of the regular group matrix is clearly $h\xi_0$. For the corresponding compound character $\phi$ we have

$$\phi(S_0) = h,$$

$$\phi(S_i) = 0 \quad (i \neq 0).$$

If $\phi = \sum \nu_i \chi^{(i)}$, then $$h\nu_i = \sum h_\rho \chi_{\rho'}^{(i)} \phi_\rho = h\chi_0^{(i)} = hf^{(i)}.$$

*The regular group matrix is equivalent to the direct sum of all the irreducible group matrices, each being repeated as many times as the degree of the character.*

Schur (2) has developed the theory of group matrices from a standpoint independent of the Frobenius algebra. He defines a group character as the set of spurs of the matrices in a matrix representation. He proves the orthogonal properties directly, and obtains the complete set of independent irreducible representations by the reduction of the regular group matrix, proving the completeness of the set so obtained by means of the orthogonal relations.

It is necessary in this development to prove specifically that reducibility implies complete reducibility, a theorem which is effectively equivalent to the theorem that a Frobenius algebra possesses no properly nilpotent element.

His method is capable of extension to continuous matrix groups, to which the method of the Frobenius algebra is not. An account is given in Chapter XI. The proof given there that reducibility implies complete reducibility is applicable to finite groups. We give now Schur's proof of the orthogonal properties for finite groups.

Let $X = \sum \xi_i M_i = [\sum \xi_i a^i_{st}]$ be an irreducible group matrix of order $f^2$. Let $X' = \sum \xi_i M'_i = [\sum \xi_i a^{i'}_{st}]$ be any independent irreducible group matrix of order $\leqslant f^2$. We suppose it to be bordered by rows and columns of zeros if the order is $< f^2$.

Firstly, if $P$ is a constant matrix such that $PX = XP$, then $P$ is a scalar multiple of the unit matrix. If $P$ is not of this form, then, transforming the matrices so that $P$ is in canonical form, it is immediately apparent that $X$ is simultaneously reduced, contrary to hypothesis.

Secondly, if $P$ is a constant matrix such that $PX = X'P$, then $P$ is non-singular and $X$ and $X'$ are equivalent.

If $P$ is non-singular, then $X$ and $X'$ are equivalent, since

$$X = P^{-1}X'P.$$

If $P$ is singular, let its rank be $r$. Then there is a matrix $Q$ of rank $f-r$ such that

$$PQ = 0.$$

Thus

$$PXQ = X'PQ = 0.$$

Now if $P$ is a singular matrix, we can find a non-singular matrix $A$ such that $P_1 = AP$ is idempotent. Transform $P$ first into canonical form. For each submatrix in the canonical form corresponding to a non-zero characteristic root there is a reciprocal. For a zero characteristic root we may multiply the submatrix by a permutation matrix which will reduce it to the form $\operatorname{diag}(0, 1^{\nu-1})$. The result follows readily. Similarly, we may find a non-singular matrix $B$ such that $Q_1 = QB$ is idempotent.

Then

$$P_1 X Q_1 = 0,$$

and transforming $P_1$ and $Q_1$ simultaneously into diagonal form, $X$ is thereby reduced, contrary to hypothesis.

Let $U = [u_{st}]$ be a fixed matrix of order $f^2$, and let

$$V = \sum M_i^{-1} U M_i, \tag{4.3; 3}$$

$$V_1 = \sum M_i'^{-1} U M_i, \tag{4.3; 4}$$

each summed for all operations of the group.

Now
$$M_j^{-1} V M_j = \sum M_j^{-1} M_i^{-1} U M_i M_j$$
$$= V,$$

since, as $M_i$ runs through all the operations of the group, so does $M_i M_j$. Thus $V$ commutes with each of the matrices $M_i$, and hence with the group matrix $X$. Since $X$ is irreducible, $V$ is a scalar multiple of the modulus.

In a similar manner

$$M_j'^{-1} V_1 M_j = \sum M_j'^{-1} M_i'^{-1} U M_i M_j$$
$$= V_1,$$
$$V_1 M_j = M_j' V_1,$$

and hence
$$V_1 = 0,$$

so that
$$\sum M_i'^{-1} U M_i = 0. \tag{4.3; 5}$$

Let $M_i^{-1} = [a_{st}^i]^{-1} = [\alpha_{st}^i]$, and let $\delta_{ii} = 1$, $\delta_{ij} = 0$, $i \neq j$. Then equation (4.3; 3) may be written

$$\sum [\alpha_{st}^i][u_{st}][a_{st}^i] = k[\delta_{st}],$$

so that
$$\sum \alpha_{pq}^{(i)} u_{qr} a_{rv}^{(i)} = k\delta_{pv},$$

where $k$ is a scalar.

Put $p = v$, and sum for all $p$. Since

$$\sum a_{rp}^{(i)} \alpha_{pq}^{(i)} = \delta_{rq},$$

we have
$$h \sum u_{qq} = kf,$$

so that
$$k = \frac{h}{f} \sum u_{qq}.$$

Hence
$${}_{iqr}\, \alpha_{pq}^{(i)} u_{qr} a_{rv}^{(i)} = \frac{h}{f} \sum u_{qq} \delta_{pv}.$$

Equating coefficients of $u_{qr}$ we have

$$\sum_i \alpha_{pq}^{(i)} a_{rv}^{(i)} = \frac{h}{f} \delta_{pv} \delta_{qr}. \tag{4.3; 6}$$

Hence if $\chi$ is the character of the group corresponding to the representation, so that

$$\chi(S_i) = \sum_q a_{qq}^{(i)},$$

then
$$\sum_i \chi(S_i)\chi(S_i^{-1}) = \sum_{iqr} \alpha_{qq}^{(i)} a_{rr}^{(i)} = \sum \frac{h}{f} \delta_{qr} \delta_{qr} = h. \tag{4.3; 7}$$

Similarly, from the equation

$$\sum M_i'^{-1} U M_i = 0$$

we obtain
$$\sum a_{pq}^{(i)} \alpha_{rv}^{(i)\prime} = 0 \tag{4.3; 8}$$

and
$$\sum_i \chi(S_i)\chi'(S_i^{-1}) = 0. \tag{4.3; 9}$$

A generalization of the orthogonal relations is easily obtained by this method. We have

$$\sum \chi(S_i)\chi(S_i^{-1}S_j) = \sum a_{pp}^{(i)} \alpha_{qr}^{(i)} a_{rq}^{(j)} = \sum \frac{h}{f} a_{pp}^{(j)} = \frac{h}{f}\chi(S_j). \tag{4.3; 10}$$

Similarly,
$$\sum \chi(S_i)\chi'(S_i^{-1}S_j) = 0. \tag{4.3; 11}$$

## 4.4. Characteristic units

An idempotent element $A$ of the Frobenius algebra of a group is called a *characteristic unit* (3).

Let $A = \sum A_i$, where $A_i$ is an element of the simple matrix algebra $\Gamma_i$. If the rank of $A_i$ is $r_i$, it can be transformed into the form

$$A_i = \operatorname{diag}(1^{r_i}, 0^{f-r_i}).$$

The characteristic unit $A$ is associated with the compound character

$$\phi = \sum r_i \chi^{(i)}.$$

A characteristic unit associated with a simple character $\chi^{(i)}$ is called a *primitive characteristic unit.* It is a primitive idempotent of the sub-algebra $\Gamma_i$.

I. *A primitive characteristic unit corresponding to the character* $\chi^{(i)}$ *has an aggregate of* $h_\rho \chi_\rho^{(i)}/h$ *elements from the class* $C_\rho$.

Since all primitive idempotents of the same simple matrix algebra have the same canonical form, they are transforms of one another, and have the same aggregate of elements from each class. But the modulus of the sub-algebra, $\epsilon_i$, may be expressed as the sum of $f^{(i)}$ primitive idempotents, and from the equation

$$h\epsilon_i = \sum f^{(i)} \chi_\rho^{(i)} C_\rho$$

the theorem follows.

Clearly, from the representation in diagonal form, any characteristic unit may be expressed as a sum of primitive characteristic units. The corresponding compound character will be the sum of the corresponding simple characters.

II. *A characteristic unit corresponding to any character* $\phi$, *simple or compound, has an aggregate of* $h_\rho \phi_\rho/h$ *elements from the class* $C_\rho$.

III. *Two characteristic units corresponding to the same character $\phi$ are transforms of one another.*

They have the same canonical form in each sub-algebra.

IV. *The product of two primitive characteristic units is either zero, nilpotent, or a multiple of a primitive characteristic unit.*

If the characteristic units belong to different sub-algebras, then the product is zero. If they belong to the same sub-algebra, the rank of each of the two corresponding matrices is unity. If the product is not zero, it must be of rank unity also. Its reduced characteristic equation must be a quadratic with one zero root, i.e.

$$x^2-\lambda x = 0.$$

If $\lambda = 0$, it is nilpotent. If $\lambda \neq 0$, it is a multiple of an idempotent.

## 4.5. The relations between the characters of a group and those of a subgroup

Let the group $H$ of order $h$ have a subgroup $G$ of order $g$. Any matrix representation of the group $H$ is clearly also a matrix representation of the group $G$, for the matrices corresponding to the subgroup $G$ are a subset of the matrices corresponding to the group $H$.

By taking the spurs of the matrices of a simple matrix representation of $H$, it follows that every simple character of $H$ is a character, simple or compound, of $G$.

Let $\chi^{(i)}$ represent any simple character of $H$, and $\phi^{(j)}$ any simple character of $G$. Then

$$\chi^{(i)} = \sum g_{ij}\,\phi^{(j)}. \tag{4.5; 1}$$

Now let $C_\rho$ be any class of $H$, of order $h_\rho$, and let $g_\rho$ of the elements of this class belong to $G$, forming the classes $C_{\rho_1}$, $C_{\rho_2}$,... of $G$, of orders $g_{\rho_1}$, $g_{\rho_2}$,... respectively.

Equation (4.5; 1) holds for each of these classes, so that

$$\chi_\rho^{(i)} = \sum g_{ij}\,\phi_{\rho_1}^{(j)} = \sum g_{ij}\,\phi_{\rho_2}^{(j)} = \dots.$$

We thus have
$$\sum g_{ij}\,g_{\rho_1}\,\phi_{\rho_1}^{(j)}\,\phi_{\rho_1'}^{(k)} = g_{\rho_1}\,\chi_\rho^{(i)}\,\phi_{\rho_1'}^{(k)},$$
$C_{\rho_1'}$ denoting the class of the inverses of the elements of the class $C_{\rho_1}$.

Summing with respect to all the classes of $G$, we obtain

$$g g_{ik} = \sum g_{\rho_1}\,\chi_\rho^{(i)}\phi_{\rho_1'}^{(k)}; \tag{4.5; 2}$$

the right-hand side being summed for all the classes of $G$, $C_\rho$ being the class of $H$ which contains the elements of the class $C_{\rho_1}$.

Hence
$$\sum_i g_{ik} g h_\sigma \chi^{(i)}_{\sigma'} = \sum g_{\rho_1} \phi^{(k)}_{\rho_1'} h_\sigma \chi^{(i)}_{\rho} \chi^{(i)}_{\sigma'}$$
$$= h \sum g_{\sigma_1} \phi^{(k)}_{\sigma_1'},$$
the last summation being confined to the classes $C_{\sigma_1}$ of $G$ which correspond to the given class $C_\sigma$ of $H$. Thus
$$\sum_i g_{ik} \chi^{(i)}_{\rho'} = \sum \frac{hg_{\rho_1}}{gh_\rho} \phi^{(k)}_{\rho_1'}.$$

Taking the inverse classes, and combining the two equations, we obtain Frobenius's formulae (4) expressing the relations between the characters of a group and those of a subgroup.
$$\chi^{(i)}_\rho = \sum_j g_{ij} \phi^{(j)}_{\rho_1},$$
$$\sum_i g_{ij} \chi^{(i)}_\rho = \sum \frac{hg_{\rho_1}}{gh_\rho} \phi^{(j)}_{\rho_1}, \qquad (4.5;3)$$
the last summation being with respect to those classes $C_{\rho_1}$, $C_{\rho_2}$,... of $G$ into which the class $C_\rho$ of $H$ separates. The coefficients $g_{ij}$ are positive integers and are the same for both equations.

The fact that $\sum hg_{\rho_1} \phi^{(j)}_{\rho_1}/gh_\rho$ is a compound character of $H$, i.e. a linear function of the characters of $H$ with *positive integral* coefficients, may be deduced otherwise from the concept of characteristic units.

A characteristic unit of a subgroup $G$ is necessarily a characteristic unit of $H$, as it is idempotent, and the group algebra of $G$ is a subalgebra of the group algebra of $H$.

But a characteristic unit of $G$ corresponding to the character $\phi^{(j)}$ has an aggregate of $g_{\rho_1} \phi^{(j)}_{\rho_1}/g$ elements from the class $C_{\rho_1}$ of $G$, and a characteristic unit of $H$ corresponding to the character $\chi^{(i)}$ has an aggregate of $h_\rho \chi^{(i)}_\rho/h$ elements from the class $C_\rho$.

Hence
$$\sum g_{\rho_1} \phi^{(j)}_{\rho_1}/g = \sum h_\rho \chi^{(i)}_\rho/h,$$
the summation on the left being with respect to the classes $C_{\rho_1}$ of $G$ which correspond to the same class $C_\rho$ of $H$, and the summation on the right being with respect to the simple characteristic units of $H$ of which the given characteristic unit of $G$ is the sum. The result follows.

## 5.1. Partitions (1)

THE theory of partitions being so intimately connected with the symmetric group, it is appropriate to commence this chapter with a brief section concerning these.

If
$$n = \lambda_1+\lambda_2+\lambda_3+\ldots+\lambda_p$$
is an equation in positive integers, then the set of numbers $(\lambda_1,\lambda_2,\ldots,\lambda_p)$, which we shall denote shortly by $(\lambda)$, is called a *partition* of $n$. No account is taken of the order of the parts, which for convenience are usually written in either ascending or descending order. Collecting parts of equal magnitude, repetitions may be indicated by the use of indices. If there are $a$ parts equal to 1, $b$ parts equal to 2, $c$ parts equal to 3, etc., the partition may be written $(1^a\, 2^b\, 3^c\ldots)$.

The number of partitions of a given integer $n$ is thus seen to be equal to the number of solutions of the equation
$$n = a+2b+3c+\ldots$$
in positive or zero integers, and hence is easily seen to be the *coefficient of $x^n$ in the expansion of*
$$1/(1-x)(1-x^2)(1-x^3)(1-x^4)\ldots \text{ to } \infty.$$

We have shown in § 3.6 that the number of classes of the symmetric group of order $n!$ is equal to the number of partitions of $n$, the class of permutations which have $a$ cycles of order 1, $b$ cycles of order 2, etc., being associated with the partition $(1^a\, 2^b\, 3^c\ldots)$.

Since there are as many characters of the symmetric group as classes, these also may be associated with partitions. The partitions associated with the characters, however, are usually written with the parts in descending order, e.g.
$$(\lambda) = (\lambda_1,\lambda_2,\ldots,\lambda_p), \qquad \lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_p.$$
The concept of *conjugate* partition is here of great importance.

The partition $(\lambda_1,\ldots,\lambda_p)$ is associated with a *graph*† which is formed as follows. Assuming that $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p$, form the graph by placing $\lambda_1$ nodes in the first row, $\lambda_2$ nodes in the second row, and so on, with finally $\lambda_p$ nodes in the $p$th row. In each row the first node is placed in the first column, the other nodes being placed consecutively one column later.

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† The 'Ferrers-Sylvester' graph of the partition. An example is given on p. 60.

If now the rows and columns of the graph are interchanged, we obtain another graph, the graph of the partition

$$(p^{\lambda_p}, (p-1)^{\lambda_{p-1}-\lambda_p}, (p-2)^{\lambda_{p-2}-\lambda_{p-1}},\ldots, 1^{\lambda_1-\lambda_2}).$$

This is called the *conjugate partition* of $(\lambda)$ and is denoted by $(\tilde{\lambda})$.

The diagonal of nodes in a graph beginning at the top left-hand corner is called the *leading diagonal*. The number of nodes in the leading diagonal is called the *rank* of the partition. Frobenius (**2**) uses the following nomenclature for partitions which brings into prominence the association between conjugate partitions.

Let $r$ be the rank of a partition, and let there be $a_i$ nodes to the right of the leading diagonal in the $i$th row, and $b_i$ nodes below the leading diagonal in the $i$th column. Then the partition is denoted by

$$\begin{pmatrix} a_1, a_2,\ldots, a_r \\ b_1, b_2,\ldots, b_r \end{pmatrix}.$$

Clearly

$$a_1 > a_2 > \ldots > a_r,$$

$$b_1 > b_2 > \ldots > b_r,$$

and

$$a_1+a_2+\ldots+a_r+b_1+b_2+\ldots+b_r+r = n.$$

The conjugate partition is

$$\begin{pmatrix} b_1, b_2,\ldots, b_r \\ a_1, a_2,\ldots, a_r \end{pmatrix}.$$

As an example, the graph of the partition $(6, 4, 3, 1^2)$ of 15 is

The nodes in the leading diagonal are in heavier type. The conjugate partition is $(5, 3^2, 2, 1^2)$. The rank is 3, and in Frobenius notation the partition is denoted by

$$\begin{pmatrix} 5, 2, 0 \\ 4, 1, 0 \end{pmatrix}.$$

This partition should not be confused with the partition $\begin{pmatrix} 5, 2 \\ 4, 1 \end{pmatrix}$, which is the partition $(6, 4, 2, 1^2)$ of 14, this being a partition of rank 2.

The partitions of $n$ are ordered as follows. Let

$$\begin{pmatrix} a_1, a_2,..., a_r \\ b_1, b_2,..., b_1 \end{pmatrix}, \qquad \begin{pmatrix} a'_1, a'_2,..., a'_{r'} \\ b'_1, b'_2,..., b'_{r'} \end{pmatrix}$$

be two partitions of $n$ expressed in Frobenius's nomenclature. We shall say that $(a_1,..., a_r) > (a'_1,..., a'_{r'})$ if the first term $a_i$ such that $a_i \neq a'_i$ is such that $a_i > a'_i$, or if $a_1 = a'_1,..., a_{r'} = a'_{r'}$, and $r > r'$.

We shall say that the partition $\begin{pmatrix} a_1,..., a_r \\ b_1,..., b_r \end{pmatrix}$ is of *lesser order* than $\begin{pmatrix} a'_1,..., a'_{r'} \\ b'_1,..., b'_{r'} \end{pmatrix}$ if either $(a_1,..., a_r) > (a'_1,..., a'_{r'})$, or if $(a_1,..., a_r) = (a'_1,..., a'_{r'})$, and $(b'_1,..., b'_{r'}) > (b_1,..., b_r)$.

We shall say that the partition $\begin{pmatrix} a_1,..., a_r \\ b_1,..., b_r \end{pmatrix}$ is of *strictly lesser order* than $\begin{pmatrix} a'_1,..., a'_{r'} \\ b'_1,..., b'_{r'} \end{pmatrix}$, if $(a_1,..., a_r) \geqslant (a'_1,..., a'_{r'})$ and also

$$(b'_1,..., b'_{r'}) \geqslant (b_1,..., b_r),$$

one of the alternative signs being an inequality.

If $(\lambda)$ and $(\mu)$ are partitions of $n$, and $(\lambda)$ is of strictly lesser order than $(\mu)$, it is clear that the conjugate partition $(\tilde{\mu})$ is of strictly lesser order than $(\tilde{\lambda})$.

If $(\lambda)$ is a partition of $n$, and another partition $(\mu)$ is obtained from it by decreasing one part and correspondingly increasing a smaller part, there is no difficulty in seeing that $(\lambda)$ is of strictly lesser order than $(\mu)$.

### Separations of a partition (3)

If the parts of a partition are divided into distinct sets, each part itself being undivided, we obtain a *separation* of the partition. Each set is called a *separate*.

Thus the partition $(1^a\, 2^b\, 3^c...)$ may be separated into

$$(1^{a_1}\, 2^{b_1}\, 3^{c_1}...), \quad (1^{a_2}\, 2^{b_2}\, 3^{c_2}...), \quad (1^{a_3}\, 2^{b_3}\, 3^{c_3}...), \quad ...,$$

provided that

$$a_1+a_2+a_3+... = a,$$
$$b_1+b_2+b_3+... = b,$$

## 5.2. Frobenius's formula for the characters of the symmetric group (4)

Frobenius gives a remarkable formula which exhibits the characteristics of the symmetric group as coefficients in a series of expansions, one for each class of the group. The proof given here follows very closely the original proof given by Frobenius. It is divided into two parts. In the first part series are given whose coefficients are compound characters, and in the second part these are separated into the simple characters.

$H$ is the symmetric group on $n$ symbols. $C_\rho$ is the class $(1^\alpha 2^\beta 3^\gamma ...)$, namely with $\alpha$ cycles of order 1, $\beta$ of order 2, etc. The order of the class $C_\rho$ is

$$h_\rho = n!/1^\alpha\alpha!\,2^\beta\beta!\,3^\gamma\gamma!.... \tag{5.2; 1}$$

Each character is associated with a partition $(\lambda) = (\lambda_1, \lambda_2,..., \lambda_p)$. Corresponding to this partition form symmetric groups $G_1, G_2,..., G_p$ respectively on distinct sets of $\lambda_1, \lambda_2,..., \lambda_p$ of the $n$ symbols. Denoting by $G_i$ also the sum of the elements of the group $G_i$, $G_i/\lambda_i!$ is clearly idempotent. Hence

$$G_1\,G_2\,...\,G_p/\lambda_1!\,\lambda_2!\,...\,\lambda_p!$$

is a characteristic unit of $H$. To obtain the corresponding compound character, we find the aggregate of elements from the class $C_\rho$.

Any element $S$ of $C_\rho$ which appears in $G_1 G_2 ... G_p$ must be the product of elements $S_1, S_2,..., S_p$ in the respective groups, and the cycles of $S$ are the sum of the cycles of $S_1, S_2,..., S_p$. Of the $\alpha$ cycles of order 1 in $S$, let $\alpha_1$ appear in $S_1$, $\alpha_2$ in $S_2$, etc., and of the $\beta$ cycles of order 2, let $\beta_1$ appear in $S_1$, $\beta_2$ in $S_2$, and so on. Then we have

$$\alpha_i+2\beta_i+3\gamma_i+... = \lambda_i,$$

$$\alpha_1+\alpha_2+\alpha_3+... = \alpha,$$

$$\beta_1+\beta_2+\beta_3+... = \beta,$$

We obtain thus a separation of the partition $(1^\alpha 2^\beta 3^\gamma ...)$ such that the separates are partitions of $\lambda_1, \lambda_2,..., \lambda_p$ respectively. For each separation the number of elements in $G_1 G_2 G_3...$ is the product of the orders of the respective classes in $G_1$, $G_2$, etc., i.e.

$$\frac{\lambda_1!}{1^{\alpha_1}\alpha_1!\,2^{\beta_1}\beta_1!...}\quad\frac{\lambda_2!}{1^{\alpha_2}\alpha_2!\,2^{\beta_2}\beta_2!...}\quad.... \tag{5.2; 2}$$

The total aggregate of elements from the class $C_\rho$ in $G_1 G_2 G_3...$ is obtained by summing this expression for all the appropriate separations of the partition.

The compound characteristic $\phi_\rho^{(\lambda)}$ is, from § 4.4, Theorem II, $h/h_\rho$

times the aggregate of elements from the class $C_\rho$ in the characteristic unit, namely

$$\phi_\rho^{(\lambda)} = n!\,\frac{1^\alpha\alpha!\,2^\beta\beta!\ldots}{n!}\,\frac{1}{\lambda_1!\,\lambda_2!\ldots}\sum\frac{\lambda_1!}{1^{\alpha_1}\alpha_1!\,2^{\beta_1}\beta_1!\ldots}\,\frac{\lambda_2!}{1^{\alpha_2}\alpha_2!\,2^{\beta_2}\beta_2!\ldots}\ldots$$

$$= \sum\frac{\alpha!}{\alpha_1!\,\alpha_2!\,\alpha_3!\ldots}\,\frac{\beta!}{\beta_1!\,\beta_2!\,\beta_3!\ldots}\ldots,$$

summed for all separations of the partition such that the separates are partitions of $\lambda_1, \lambda_2,\ldots$.

If $x_1, x_2,\ldots, x_m$ are independent variables with $m$ not less than $p$, the number of parts in the partition $(\lambda)$, then the above sum is the coefficient of

$$x_1^{\lambda_1}\,x_2^{\lambda_2}\ldots x_p^{\lambda_p}$$

in the product

$$(x_1+x_2+\ldots+x_m)^\alpha(x_1^2+x_2^2+\ldots+x_m^2)^\beta(x_1^3+x_2^3+\ldots+x_m^3)^\gamma\ldots.$$

Let $S_r = \sum x_i^r$, and corresponding to the class $C_\rho$ form the product

$$S_\rho = S_1^\alpha\,S_2^\beta\,S_3^\gamma\ldots. \tag{5.2; 3}$$

Then

$$S_\rho = \sum\phi_\rho^{(\lambda)}\,x_1^{\lambda_1}x_2^{\lambda_2}\ldots x_p^{\lambda_p}, \tag{5.2; 4}$$

the summation being with respect to all partitions $(\lambda)$ of $n$, and also, as in the usual nomenclature $\sum x_1^{\lambda_1}\ldots x_p^{\lambda_p}$ for monomial symmetric functions, with respect to all permutations of the suffixes without duplication for the interchange of suffixes which correspond to equal indices. $\phi^{(\lambda)}$ is the compound character of the symmetric group corresponding to the characteristic unit $G_1\,G_2\,G_3\ldots/\lambda_1!\,\lambda_2!\,\lambda_3!\ldots$.

If $m \geqslant n$, the power sums $S_1, S_2,\ldots, S_n$ are algebraically independent, and the products $S_\rho$ linearly independent. The matrix of coefficients $\phi_\rho^{(\lambda)}$ corresponding to the $q$ classes and $q$ partitions is non-singular, so that the $q$ characters $\phi^{(\lambda)}$ are linearly independent. The $q$ simple characters must be linearly dependent upon these.

Before proceeding to the separation of the simple characters in the general case we illustrate by an example for the case of the symmetric group of order 3!, and deduce the simple characters by the use of the orthogonal properties of the characters.

$$\begin{aligned} S_1^3 &= \sum x_1^3+3\sum x_1^2x_2+6\sum x_1\,x_2\,x_3,\\ S_1\,S_2 &= \sum x_1^3+\sum x_1^2x_2,\\ S_3 &= \sum x_1^3. \end{aligned}$$

Hence the compound characters are

| Class . . | $(1^3)$ | $(2\,1)$ | $(3)$ |
|---|---|---|---|
| Order . . | 1 | 3 | 2 |
| $\phi^{(3)}$ | 1 | 1 | 1 |
| $\phi^{(21)}$ | 3 | 1 | 0 |
| $\phi^{(1^3)}$ | 6 | 0 | 0 |

$\phi^{(3)} = \chi^{(3)}$ is clearly simple. We have

$$\sum h_\rho \chi_\rho^{(3)} \phi_\rho^{(21)} = 6,$$

so that $\phi^{(21)}$ includes the character $\chi^{(3)}$ once. $\chi^{(21)} = \phi^{(21)} - \chi^{(3)}$ is simple, for we have

$$\sum h_\rho \chi_\rho^{(21)2} = 6.$$

Similarly,

$$\sum h_\rho \chi_\rho^{(3)} \phi_\rho^{(1^3)} = 6,$$

$$\sum h_\rho \chi_\rho^{(21)} \phi_\rho^{(1^3)} = 12,$$

and $\chi^{(1^3)} = \phi^{(1^3)} - \chi^{(3)} - 2\chi^{(21)}$ is simple, for

$$\sum h_\rho \chi_\rho^{(1^3)2} = 6.$$

Hence we obtain the table of simple characters.

| Class . . | $(1^3)$ | $(2\,1)$ | $(3)$ |
|---|---|---|---|
| Order . . | 1 | 3 | 2 |
| $\chi^{(3)}$ | 1 | 1 | 1 |
| $\chi^{(21)}$ | 2 | 0 | $-1$ |
| $\chi^{(1^3)}$ | 1 | $-1$ | 1 |

This procedure is not practicable except for the groups of small orders. Fortunately, Frobenius has crowned his work by a formula which separates the simple characters in the general case.

Let

$$\Delta(x_1,\ldots,x_m) = \prod (x_r - x_s) \qquad (r < s)$$
$$= \sum \pm x_1^{m-1} x_2^{m-2} \ldots x_{m-1}.$$

The expression is an alternating function of the variables, that is, the effect of an interchange of two of the variables is to leave the expression unaltered in magnitude, but changed in sign. One term only is given throughout for such alternating functions, with a summation sign, it being understood that all permutations of the suffixes are taken, and of the alternative signs, the negative sign is taken for a negative permutation.

The product $S_\rho \Delta(x_1,\ldots,x_m)$ is also an alternating function. In the development the coefficient of any term with two equal indices is zero, since the interchange of the suffixes corresponding to two equal indices would change the sign of the coefficient, but also leave it unaltered. Thus, arranging the indices in descending order, we may put

$$S_\rho\,\Delta(x_1,\dots,x_m) = \sum_\lambda \pm\psi_\rho^{(\lambda)} x_1^{\lambda_1+m-1}x_2^{\lambda_2+m-2}\dots x_m^{\lambda_m}, \qquad (5.2;5)$$

with

$$\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_m,$$

and

$$\lambda_1+\lambda_2+\dots+\lambda_m = n.$$

The summation as before is with respect to all permutations of the suffixes, with the alternative sign, and also with respect to all partitions $(\lambda)$ of $n$.

Each coefficient $\psi_\rho^{(\lambda)}$ is a linear function of the compound characteristics $\phi_\rho^{(\lambda)}$, with positive or negative integral coefficients. Hence $\psi^{(\lambda)}$ is a linear function of the simple characters with positive or negative integral coefficients.

If

$$\psi_\rho^{(\lambda)} = \sum r_{(\mu)}^{(\lambda)}\,\chi_\rho^{(\mu)},$$

then

$$\sum h_\rho\,\psi_\rho^{(\lambda)2} = h\sum_{(\mu)} r_{(\mu)}^{(\lambda)2}.$$

The remainder of the proof consists in showing that the value of this expression is $h$, so that $r_{(\mu)}^{(\lambda)} = \pm 1$ for one partition $(\mu)$, and is zero for all others. Since it will be apparent that $\psi_0^{(\lambda)}$ is positive, it will follow that $\psi^{(\lambda)}$ is a simple character.

Take a second set of $m$ independent variables $y_1, y_2,\dots, y_m$. Let $Z_r = \sum_i y_i^r$, and put $Z_\rho = Z_1^\alpha Z_2^\beta Z_3^\gamma\dots$.

Consider the sum

$$\sum\frac{h_\rho}{h}S_\rho Z_\rho = \sum\frac{h_\rho}{h}S_1^\alpha S_2^\beta S_3^\gamma\dots Z_1^\alpha Z_2^\beta Z_3^\gamma\dots$$

$$= \sum\frac{1}{1^\alpha\alpha!}(S_1 Z_1)^\alpha\frac{1}{2^\beta\beta!}(S_2 Z_2)^\beta\dots,$$

the last expression being summed for all solutions of

$$\alpha+2\beta+3\gamma+\dots = n.$$

Summed for all values of $n$ from 0 to $\infty$, with an appropriate restriction on the variables to ensure convergence, the value of this sum is

$$\exp[S_1 Z_1+\tfrac{1}{2}S_2 Z_2+\tfrac{1}{3}S_3 Z_3+\dots].$$

Substituting $S_r = \sum x_i^r$, $Z_r = \sum y_i^r$ and multiplying out, the value of the bracket becomes

$$-\log(1-x_1y_1)-\log(1-x_1y_2)-\ldots-\log(1-x_my_m).$$

It follows that the given sum is the reciprocal of

$$(1-x_1y_1)(1-x_1y_2)\ldots(1-x_my_m).$$

LEMMA.

$$1-x_sy_t \quad = \frac{\Delta(x_r)\Delta(y_r)}{\prod(1-x_py_q)}.$$

On multiplying each $p$th row of this determinant by $\prod_q (1-x_py_q)$ the determinant becomes of the form $|f_t(x_s)|$, where $f_t(x)$ is a polynomial of degree $n-1$ whose coefficients involve the $y_q$'s but not the $x_p$'s. Clearly $\Delta(x_r) \equiv \Delta(x_1, x_2,\ldots, x_m)$ is a factor, for the determinant is an alternating function of the $x_r$'s. Further, since elementary transformations will reduce the degrees of consecutive columns to $n-1, n-2, n-3$, etc., the degree of the quotient by this factor is zero in the $x_r$'s. If the further factor $\Delta(y_r)$ is removed, the quotient is independent of the $x_r$'s and of the $y_r$'s, and the proof of the lemma follows readily, the numerical factor being found by comparison of a coefficient.

But $$1/(1-x_sy_t) = \sum_0^\infty x_s^\nu y_t^\nu.$$

Hence $$|1-x_sy_t| = \sum \pm x_{e_1}^{\nu_1}y_1^{\nu_1}x_{e_2}^{\nu_2}y_2^{\nu_2}\ldots x_{e_m}^{\nu_m}y_m^{\nu_m}.$$

The indices $\nu_1,\ldots, \nu_m$ are necessarily distinct, since we deal with an alternating function of both sets of variables. If we assume that the indices are in descending order, the summation must be taken for all permutations of both sets of suffixes, the minus sign being taken if one permutation only is negative.

Putting $\nu_r = \lambda_r+m-r$, we have

$$\sum \frac{h_\rho}{h} S_\rho Z_\rho \Delta(x_r)\Delta(y_r)$$

$$= \sum \frac{1}{1^\alpha\alpha!} S_1^\alpha Z_1^\alpha \frac{1}{2^\beta\beta!} S_2^\beta Z_2^\beta \ldots \Delta(x_r)\Delta(y_r)$$

$$= \sum \pm x_e^{\lambda_1+m-1}y_1^{\lambda_1+m-1}x_{e_2}^{\lambda_2+m-2}y_2^{\lambda_2+m-2}\ldots x_{e_m}^{\lambda_m}y_m^{\lambda_m}, \quad (5.2;6)$$

with $$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \ldots \geqslant \lambda_m.$$

The summation on the left is with respect to all partitions of all positive integers, i.e. with respect to all classes of all the symmetric groups.

Now substitute in (5.2; 6) the expression for $S_\rho \Delta(x_r)$ given in (5.2; 5), and a similar expression for $Z_\rho \Delta(y_r)$. Equating coefficients it follows that

$$\begin{aligned} \sum \frac{h_\rho}{h} \psi_\rho^{(\lambda)} \psi_\rho^{(\mu)} &= 0 \quad ((\lambda) \neq (\mu)), \\ \sum \frac{h_\rho}{h} \psi_\rho^{(\lambda)2} &= 1. \end{aligned} \qquad (5.2;7)$$

Hence $\psi^{(\lambda)}$ is equal to $\pm$ a simple character. We shall see in the next section that $\psi_0^{(\lambda)}$ is equal to the number of lattice permutations of $x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_m^{\lambda_m}$, which is necessarily positive, so that we shall assume now that the sign is positive.

Equations (5.2; 7) also show that the $p$ characters $\psi^{(\lambda)}$ corresponding to the $p$ partitions $(\lambda)$ are distinct characters, and these are the complete set of characters of the group. Putting $\psi_\rho^{(\lambda)} = \chi_\rho^{(\lambda)}$, we obtain

$$S_\rho \Delta(x_r) = \sum \pm \chi_\rho^{(\lambda)} x_1^{\lambda_1+m-1} x_2^{\lambda_2+m-2} \ldots x_m^{\lambda_m}. \qquad (5.2;8)$$

This is Frobenius's formula for the characters $\chi^{(\lambda)}$ of the symmetric group.

## 5.3. Characters and lattices (5)

Let $(\lambda) \equiv (\lambda_1, \lambda_2, \ldots, \lambda_p)$, $(\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p)$, be a partition of $n$. Consider the permutations of the $n$ factors of the product

$$x_1^{\lambda_1} x_2^{\lambda_2} \ldots x_p^{\lambda_p}.$$

If amongst the first $r$ terms of any such permutation the number of times $x_1$ occurs $\geqslant$ the number of times $x_2$ occurs $\geqslant$ the number of times $x_3$ occurs, etc., for all values of $r$, this is called a *lattice permutation* (6).

As an example, the lattice permutations of $x_1^3 x_2 x_3$ are

$$\begin{array}{lll} x_1^3 x_2 x_3, & x_1^2 x_2 x_1 x_3, & x_1^2 x_2 x_3 x_1, \\ x_1 x_2 x_1^2 x_3, & x_1 x_2 x_1 x_3 x_1, & x_1 x_2 x_3 x_1^2. \end{array}$$

Now consider graphs which consist of nodes arranged in rows and columns. A graph is said to be *regular* if in any row which contains, say, $r$ nodes, these occur in the first $r$ columns, and in any column which contains, say, $r$ nodes, these occur in the first $r$ rows. Thus the graph of a partition is a regular graph, and, conversely, every regular graph is the graph of a partition.

If to a regular graph one node is added so that the resulting graph is also regular, this is called a *regular application of a node*.

*The number of ways of building the graph of the partition* $(\lambda)$ *of* $n$ *by* $n$ *regular applications of nodes is equal to the number of lattice permutations of* $x_1^{\lambda_1}x_2^{\lambda_2}x_3^{\lambda_3}\dots x_p^{\lambda_p}$.

Corresponding to any lattice permutation, commencing at the first term, whenever $x_r$ occurs place a node in the $r$th row, in the first vacant column. Since the permutation is lattice the graph is regular at every stage. Conversely, for each method of building the graph the corresponding permutation is lattice.

By way of illustration, there are six ways of building the graph of the partition $(3\,1^2)$, which correspond to the six lattice permutations of $x_1^3x_2x_3$ given above. These are, using numbers instead of nodes so as to indicate the order of application,

$$\begin{array}{lll} 1,\ 2,\ 3 & 1,\ 2,\ 4 & 1,\ 2,\ 5 \\ 4 & 3 & 3 \\ 5 & 5 & 4 \end{array}$$

$$\begin{array}{lll} 1,\ 3,\ 4 & 1,\ 3,\ 5 & 1,\ 4,\ 5 \\ 2 & 2 & 2 \\ 5 & 4 & 3 \end{array}$$

An immediate corollary of the theorem is that if $(\mu) \equiv (\mu_1, \mu_2,\dots, \mu_q)$ is the partition conjugate to $(\lambda)$, then the number of lattice permutations of $x_1^{\mu_1}x_2^{\mu_2}\dots x_q^{\mu_q}$ is equal to the number of lattice permutations of $x_1^{\lambda_1}x_2^{\lambda_2}\dots x_p^{\lambda_p}$. The proof is obtained by interchanging the rows and columns of the graph.

I. *If* $(\lambda) = (\lambda_1,\dots,\lambda_p)$ *is a partition of* $n$, *then* $f^{(\lambda)} = \chi_0^{(\lambda)}$ *is equal to the number of lattice permutations of* $x_1^{\lambda_1}x_2^{\lambda_2}\dots x_p^{\lambda_p}$, *which is equal to the number of ways of building the graph of the partition* $(\lambda)$ *by* $n$ *regular applications of nodes.*

We use Frobenius's formula for the characters of the symmetric group, and pick out the coefficient of $x_1^{\lambda_1+p-1}x_2^{\lambda_2+p-2}\dots x_p^{\lambda_p}$ in the product $\Delta(x_1,x_2,\dots,x_p)S_1^n$.

We take the expression $\Delta(x_1,\dots,x_p) = \sum \pm x_1^{p-1}x_2^{p-2}\dots x_{p-1}$ and multiply by $S_1 = x_1+x_2+\dots+x_p$, $n$ times. At any step, since the function is alternating, the coefficient of any term in which two indices are equal is zero.

Again, since at each stage we multiply by a linear function of the $x_r$'s, the order of magnitude of the indices in any term can never change unless we pass through a stage where two indices are equal. At this stage, however, as we have seen, the coefficient is zero.

Hence, to pick out the required coefficient we need consider those terms only in which the order of the indices is decreasing. We consider thus the coefficients at each step of the terms

$$x_1^{p-1}x_2^{p-2}\dots x_{p-1}.x_1^{\mu_1}x_2^{\mu_2}\dots \quad (\mu_1 \geqslant \mu_2 \geqslant \dots).$$

We obtain the terms at the next step by multiplying these terms by $(x_1+x_2+\dots+x_p)$ and consider those terms only which still satisfy

$$\mu_1 \geqslant \mu_2 \geqslant \dots.$$

It is clear that the $f^{(\lambda)}$ terms $x_1^{\lambda_1+p-1}x_2^{\lambda_2+p-2}\dots x_p^{\lambda_p}$ in the product

$$\Delta(x_1,\dots,x_p)S_1^n$$

must be obtained by this process, and for each term we obtain a lattice permutation of

$$x_1^{\lambda_1}x_2^{\lambda_2}\dots x_p^{\lambda_p},$$

and the theorem is proved.

Now for a class containing a cycle of order $r$, $S_1^r$ must be replaced by $S_r$. The order of magnitude of the indices may now change without passing through a stage in which two indices are equal.

Consider the term

$$(x_1^{p-1}x_2^{p-2}\dots x_{p-1})x_1^{\alpha_1}x_2^{\alpha_2}\dots x_q^{\alpha_q}x_{q+1}^{\alpha_q+1}\dots x_{q+i}^{\alpha_q+i}\dots \quad (\alpha_1 \geqslant \alpha_2 \geqslant \dots \geqslant \alpha_p).$$

Multiplying by $S_r$ we consider that term obtained by multiplying by $x_{q+i}^r$. If two indices become equal, the coefficient will be zero as before; hence we assume them all to be different. Suppose

$$\alpha_q+p-q > \alpha_{q+i}+p-q-i+r > \alpha_{q+1}+p-q-1.$$

We obtain the term

$$(x_1^{p-1}\dots x_q^{p-q}x_{q+i}^{p-q-1}x_{q+1}^{p-q-2}\dots x_{p-1})x_1^{\alpha_1}\dots x_q^{\alpha_q}x_{q+i}^{\alpha_{q+i}+r-i+1}x_{q+1}^{\alpha_{q+1}+1}\dots$$
$$\dots x_{q+i-1}^{\alpha_{q+i-1}+1}x_{q+i+1}^{\alpha_{q+i+1}}\dots.$$

There will be a corresponding term in which the order of the suffixes is the natural order, a negative sign being attached if $i$ is even. This is the term considered at the next stage.

We express the above in terms of lattices as follows.

The addition of $r$ nodes to a regular graph is called a *regular application of r nodes* if the nodes are added to any row until they are exhausted, or until the number of nodes in this row exceeds the number in the preceding row by one, the nodes being then added to the preceding row according to the same rule, and so on until the $r$ nodes are exhausted, provided that the final graph obtained is regular. If the number of rows involved is even it is called a *negative* application, if odd, a *positive* application.

II. *If* $(\lambda) \equiv (\lambda_1, \lambda_2, ..., \lambda_p)$ *is a partition of* $n$, *and* $\rho$ *denotes the class of the symmetric group with cycles of orders* $a_1$, $a_2$, $a_3, ...,$ *then* $\chi_\rho^{(\lambda)}$ *is obtained from the number of methods of building the graph of the partition* $(\lambda)$ *by consecutive regular applications of* $a_1$, $a_2$, $a_3, ...$ *nodes, by subtracting the number of methods which contain an odd number of negative applications from the number of methods which contain an even number of negative applications.*

As an example of the practical use of the theorem we find the value of the character $\chi^{(16, 2^2)}$ for the class of the symmetric group of order 20!, which contains one cycle of order 15, two cycles of order 2, and one of order 1. The possible graphs, using numbers in the place of nodes to indicate at which step the nodes are added, are the following:

$$\begin{array}{llll}
1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4 & & 1,1,1,1,1,1,1,1,1,1,1,1,1,1,1,4 \\
2,2 & & 2,3 \\
3,3 & & 2,3 \\
 & & \\
1,1,1,1,1,1,1,1,1,1,1,1,1,2,2,4 & & 1,1,1,1,1,1,1,1,1,1,1,1,1,3,3,4 \\
1,3 & & 1,2 \\
1,3 & & 1,2
\end{array}$$

The first and second graphs contain respectively 0 and 2 negative applications, and each contributes $+1$ to the characteristic. The last two graphs each contain one negative application, and these each contribute $-1$. Hence for this class

$$\chi_\rho^{(16, 2^2)} = 1+1-1-1 = 0.$$

Various results follow immediately from Theorem II concerning the characters of the symmetric group.

III. $$\chi_\rho^{(n)} = 1 \quad \textit{for every class.}$$

There is but one row and one method of building the graph.

IV. $$\chi_\rho^{(1^n)} = +1 \quad \textit{for a positive class,}$$
$$= -1 \quad \textit{for a negative class.}$$

There is but one column and one method of building the graph. Each cycle of even order contributes a factor $-1$.

Now interchanging the rows and columns of a graph, it is clear that to each regular application of $r$ nodes to a given graph there corresponds a regular application of $r$ nodes to the conjugate graph, the role of the rows and columns being interchangeable in the definition of regular application, save that the order of application would be reversed. Further, in any regular application of $r$ nodes the number of rows involved plus the number of columns involved is equal to $r+1$. Hence, if $r$ is odd, the application of $r$ nodes to a graph and the corresponding application of $r$ nodes to the conjugate graph are positive or negative applications together. If $r$ is even, one is positive and the other negative. Corresponding to a given class $\rho$, for every method of building the graph of a partition $(\lambda)$ there is a method of building the graph of the conjugate partition $(\mu)$. For a positive class the methods will contribute $+1$ or $-1$ together, and for a negative class one will contribute $+1$ and the other $-1$.

V. *If* $(\mu)$ *is the partition conjugate to* $(\lambda)$,

$$\chi_\rho^{(\mu)} = \chi_\rho^{(\lambda)} \chi_\rho^{(1^n)}.$$

## 5.4. Primitive characteristic units and Young tableaux

Let $\phi_\rho^{(\lambda)}$ be the coefficient of $x_1^{\lambda_1} x_2^{\lambda_2} \dots x_p^{\lambda_p}$ $(\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_p)$ in the product $S_\rho$ as defined by the equation (5.2; 3). Now the simple characteristic $\chi_\rho^{(\lambda)}$ is the coefficient of $x_1^{\lambda_1+m-1} x_2^{\lambda_2+m-2} \dots x_m^{\lambda_m}$ in the product $S_\rho \Delta(x_1,\dots,x_m) = S_\rho \sum \pm x_1^{m-1} x_2^{m-2} \dots x_{m-1}$.

It is clear that $\chi^{(\lambda)}$ is a linear function of the compound characters $\phi^{(\lambda)}$ of the form

$$\chi^{(\lambda)} = \phi^{(\lambda)} + \sum k_{\lambda\mu} \phi^{(\mu)},$$

in which the coefficient of $\phi^{(\lambda)}$ is unity, and only such partitions $(\mu)$ occur as are obtained from $(\lambda)$ by increasing larger parts and correspondingly decreasing smaller parts. From § 5.1 the partition $(\mu)$ is of strictly lesser order than the partitions $(\lambda)$.

Hence, solving these equations for the $\phi^{(\lambda)}$, we have

$$\phi^{(\lambda)} = \chi^{(\lambda)} + \sum k'_{\lambda\mu} \chi^{(\mu)}, \tag{5.4; 1}$$

where again the partition $(\mu)$ is of strictly lesser order than $(\lambda)$.

Denoting a conjugate partition by a tilde, we have also

$$\phi^{(\tilde{\lambda})} = \chi^{(\tilde{\lambda})} + \sum k''_{\lambda\nu} \chi^{(\tilde{\nu})} \tag{5.4; 2}$$

in which the partition $(\tilde{\nu})$ is of strictly lesser order than the partitions $(\tilde{\lambda})$.

It follows from § 5.1 that $(\lambda)$ is of strictly lesser order than $(\nu)$. Hence for every pair of partitions $(\lambda)$ and $(\mu)$ either $k'_{\lambda\mu}$ or $k''_{\lambda\mu}$ is zero.

We now proceed to construct a series of primitive characteristic units corresponding to each invariant sub-algebra of the Frobenius algebra of the symmetric group.

We shall use the term *symmetric group on r symbols* to denote the sum of the group elements of the symmetric group which permutes these symbols; and the term *negative symmetric group* for the same thing with a minus sign attached to each negative permutation.

Now in the graph of the partition $(\lambda)$ of $n$, replace the $n$ nodes by the symbols $\alpha_1, \alpha_2, \ldots, \alpha_n$ taken in any order. The result we shall call a *Young tableau*. They were first used by Young† in his papers on Quantitative Substitutional Analysis. Frobenius‡ adapted the results to his study of Group Characters.

Now take the product $P$ of the symmetric groups on the $\lambda_i$ symbols in each $i$th row of the Young tableau. From § 5.2, $P/\lambda_1!\,\lambda_2!\ldots\lambda_p!$ is a characteristic unit corresponding to the compound character

$$\phi^{(\lambda)} = \chi^{(\lambda)} + \sum k'_{\lambda\mu}\,\chi^{(\mu)}.$$

If $(\tilde{\lambda}) \equiv (\tilde{\lambda}_1, \tilde{\lambda}_2, \ldots, \tilde{\lambda}_q)$ is the partition conjugate to $(\lambda)$, then there are $\tilde{\lambda}_i$ symbols in the $i$th column. The product of the symmetric groups on the symbols in the columns is a multiple of a characteristic unit corresponding to the compound character

$$\phi^{(\lambda)} = \chi^{(\tilde{\lambda})} + \sum k''_{\lambda\nu}\,\chi^{(\tilde{\nu})}.$$

Hence, if $N$ is the product of the negative symmetric groups on the symbols in the columns, $N/\tilde{\lambda}_1!\,\tilde{\lambda}_2!\ldots\tilde{\lambda}_q!$ is a characteristic corresponding to the compound character

$$\widetilde{\phi^{(\tilde{\lambda})}} = \chi^{(\lambda)} + \sum k''_{\lambda\nu}\,\chi^{(\nu)},$$

and none of the characters $\chi^{(\nu)}$ coincides with a character $\chi^{(\mu)}$.

Thus the product $$PN/\lambda_1!\,\lambda_2!\ldots\lambda_p!\,\tilde{\lambda}_1!\ldots\tilde{\lambda}_q!$$

is equal to the product of two primitive characteristic units each of which corresponds to the simple character $\chi^{(\lambda)}$, since the corresponding sub-algebra of the Frobenius algebra is the only sub-algebra common to the two characteristic units.

The product $PN$ is, from § 4.4, Theorem IV, either zero, nilpotent, or a multiple of a primitive characteristic unit. It is clear that the coefficient of the identical element in $PN$ is unity, since the quantities $P$ and $N$ have the identical element and no other group element in common. Hence the trace of $PN$ is not zero, and $PN$ is neither zero nor nilpotent but a multiple of a primitive characteristic unit corresponding to the character $\chi^{(\lambda)}$.

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† Young (74), Part I.

‡ Frobenius (35).

From § 4.4, Theorem I, a primitive characteristic unit corresponding to the character $\chi^{(\lambda)}$ has an aggregate of $h_\rho \chi_\rho^{(\lambda)}/h$ elements from the class $C_\rho$, so that the coefficient of the identical element is

$$\chi_0^{(\lambda)}/h = f^{(\lambda)}/h.$$

Hence the required primitive characteristic unit is

$$f^{(\lambda)}PN/h. \tag{5.4;3}$$

I. *If $P$ is the product of the symmetric groups on the symbols in each row, and $N$ the product of the negative symmetric groups on the symbols in each column of a Young tableau corresponding to the partition $(\lambda)$, then*

$$f^{(\lambda)}PN/h$$

*is a primitive characteristic unit corresponding to the character $\chi^{(\lambda)}$.*

The order of the factors $P$ and $N$ could be reversed; $f^{(\lambda)}NP/h$ is also a primitive characteristic unit.

As an example, for the symmetric group on four symbols $\alpha$, $\beta$, $\gamma$, $\delta$, corresponding to the partition $(3, 1)$, there is a Young tableau

$$\begin{pmatrix} \alpha, \beta, \gamma \\ \delta \end{pmatrix}.$$

We have

$$\begin{aligned} P &= I+(\alpha\beta)+(\alpha\gamma)+(\beta\gamma)+(\alpha\beta\gamma)+(\alpha\gamma\beta), \\ N &= I-(\alpha\delta), \\ PN &= I+(\alpha\beta)+(\alpha\gamma)+(\beta\gamma)+(\alpha\beta\gamma)+(\alpha\gamma\beta)-(\alpha\delta)- \\ &\qquad -(\alpha\beta\delta)-(\alpha\gamma\delta)-(\beta\gamma)(\alpha\delta)-(\alpha\beta\gamma\delta)-(\alpha\gamma\beta\delta). \end{aligned}$$

Since $\chi_0^{(31)} = 3$, and $h = 24$, the primitive characteristic unit is

$$\tfrac{3}{24}PN = \tfrac{1}{8}PN.$$

The aggregates of elements from the classes

$$(1^4),\quad (1^2\,2),\quad (1\,3),\quad (4),\quad (2^2)$$

in the characteristic unit are respectively

$$\tfrac{1}{8},\quad \tfrac{1}{4},\quad 0,\quad -\tfrac{1}{4},\quad -\tfrac{1}{8}.$$

The orders of the classes are

$$1,\quad 6,\quad 8,\quad 6,\quad 3.$$

Hence by multiplying the aggregates by $h/h_\rho$ we obtain the simple character $\chi^{(31)}$ which takes the values

$$3,\quad 1,\quad 0,\quad -1,\quad -1.$$

The characteristic unit $f^{(\lambda)}PN/h$ we shall call a *Young tableau unit.* Corresponding to a tableau $A$, the tableau unit will be denoted by $(A)$. In future the factor $f^{(\lambda)}/h$ will be understood to be combined with the $P$, so that we shall write $(A) = PN$.

II. *The product of two Young tableau units corresponding to different partitions is zero.*

For the characteristic units belong to different invariant sub-algebras.

The Young tableau units have one fundamental property which we proceed to obtain.

Let $Z_1$ denote the symmetric group on $p$ symbols which include $\alpha$ and $\beta$. Clearly $Z_1 = Z_1(\alpha\beta) = (\alpha\beta)Z_1$, so that in particular

$$Z_1 = \tfrac{1}{2}Z_1[I+(\alpha\beta)] = \tfrac{1}{2}[I+(\alpha\beta)]Z_1.$$

Let $Y_1$ denote the negative symmetric group on $q$ symbols which also include $\alpha$ and $\beta$. Then, similarly,

$$Y_1 = \tfrac{1}{2}Y_1[I-(\alpha\beta)] = \tfrac{1}{2}[I-(\alpha\beta)]Y_1.$$

Hence $$Z_1Y_1 = \tfrac{1}{4}Z_1[I+(\alpha\beta)][I-(\alpha\beta)]Y_1 = 0.$$

The following theorem follows immediately.

III. *If $(A_i) = P_iN_i$ and $(A_j) = P_jN_j$ are two Young tableau units such that two symbols in the same row in $A_j$ are in the same column in $A_i$, then $N_iP_j$, $P_jN_i$, and $(A_i)(A_j)$ are zero.*

Following the method of proof of this theorem, an alternative proof of Theorem II can be obtained, and the whole theory can be developed without reference to Frobenius's formula for the characters of the symmetric group.

There are $n!$ tableaux that can be built corresponding to any partition $(\lambda)$. Of these we choose $f^{(\lambda)}$ tableaux, which we call *standard tableaux.*

*If any assigned order is given to the $n$ symbols on which the symmetric group operates, then a tableau in which the order of the symbols in each row and in each column follows the assigned order is called a standard tableau.*

If the tableau is built by adding the symbols in their assigned order, it is obvious from § 5.3, Theorem I, that—

IV. *Corresponding to a partition* $(\lambda)$ *there are exactly* $f^{(\lambda)}$ *standard tableaux.*

These $f^{(\lambda)}$ tableaux are ordered in the following manner. Let $A$ and $B$ represent two standard tableaux such that the symbols in the $i$th row and $j$th column are $a_{ij}$ and $b_{ij}$ respectively. Consider the sequences

$$a_{11},\ a_{12},\ \ a_{1\lambda_1},\ \ a_{21},\ldots,\ a_{2\lambda_2},\ \ a_{31},\ \ a_{p\lambda_p};$$
$$b_{11},\ b_{12},\ \ b_{1\lambda_1},\ \ b_{21},\ldots,\ b_{2\lambda_2},\ \ b_{31},\ldots,\ b_{p\lambda_p}.$$

Let $a_{ij}$ be the first symbol in the first sequence such that $a_{ij} \neq b_{ij}$. Then if $a_{ij}$ precedes $b_{ij}$ in the assigned order, the tableau $A$ is said to precede the tableau $B$.

V. *If* $(A)$, $(B)$ *are two standard tableau units corresponding to the same partition, and* $A$ *precedes* $B$, *then* $(B)(A) = 0$.

Let $a_{ij}$ and $b_{ij}$ be the elements in the $i$th row, $j$th column of $A$ and $B$ respectively, and let $a_{ij}$ be the first symbol in the sequence $a_{11},\ldots, a_{1\lambda_1}, a_{21},\ldots, a_{p\lambda_p}$ such that $a_{ij} \neq b_{ij}$. Thus $a_{ij}$ precedes $b_{ij}$. The symbols in the first $(i-1)$ rows and the first $(j-1)$ symbols in the $i$th row of the two tableaux are identical. Hence the symbol $a_{ij}$ must appear in $B$ in a later row than the $i$th. Since $a_{ij}$ precedes $b_{ij}$ it must appear in an earlier column than the $j$th. Suppose, then, that

$$a_{ij} = b_{kl} \quad (i < k,\ j > l).$$

Then the elements $a_{il} = b_{il}$, $a_{ij} = b_{kl}$ appear in the same row in $A$, and in the same column in $B$, so that $(B)(A) = 0$.

Let the $f^{(\lambda)}$ standard tableaux corresponding to the partition $(\lambda)$ be $A_1, A_2,\ldots, A_{f^{(\lambda)}}$, where $A_i$ precedes $A_j$ if $i < j$, and let $(A_i) = P_i N_i$. Thus for $i < j$, $P_j N_j P_i N_i = 0$, and in particular $N_j P_i = 0$.

It usually happens that there is also at least one pair of symbols in the same column in $A_i$ and in the same row in $A_j$. If this is the case $N_i P_j = 0$ also.

If this is true for every pair of tableaux, a simple matrix representation of the sub-algebra is easily obtained.

Let $\sigma_{ij}$ be the permutation which transforms the tableau $A_i$ into the tableau $A_j$, so that

$$P_i \sigma_{ij} = \sigma_{ij} P_j,$$
$$N_i \sigma_{ij} = \sigma_{ij} N_j.$$

Also
$$\sigma_{ij}\sigma_{jk} = \sigma_{ik}.$$

Now put
$$\left.\begin{aligned} e_{ii} &= P_i N_i, \\ e_{ij} &= P_i \sigma_{ij} N_j. \end{aligned}\right\} \tag{5.4; 4}$$

Then
$$\begin{aligned} e_{ij}e_{jk} &= P_i \sigma_{ij} N_j P_j \sigma_{jk} N_k \\ &= \sigma_{ij}\sigma_{jk} P_k N_k P_k N_k \\ &= \sigma_{ik} P_k N_k \\ &= P_i \sigma_{ik} N_k \\ &= e_{ik}, \\ e_{ij}e_{kl} &= P_i \sigma_{ij} N_j P_k \sigma_{kl} N_l \\ &= 0. \end{aligned}$$

The simple matrix representation is evident.

Unfortunately some cases occur in which there is no pair of elements in the same row in $A_j$ and in the same column in $A_i$. Thus for the partition (3 2) for the symmetric group on five symbols $\alpha_1$, $\alpha_2$, $\alpha_3$, $\alpha_4$, $\alpha_5$, taken in that order,

$$A_1 = \begin{pmatrix} \alpha_1 & \alpha_2 & \alpha_3 \\ \alpha_4 & \alpha_5 & \end{pmatrix},$$
$$A_5 = \begin{pmatrix} \alpha_1 & \alpha_3 & \alpha_5 \\ \alpha_2 & \alpha_4 & \end{pmatrix}.$$

We have $N_5 P_1 = 0$, but $N_1 P_5 \neq 0$. We cannot choose $P_1 N_1$ and $P_5 N_5$ as diagonal elements in our simple matrix representation because $P_1 N_1 P_5 N_5 \neq 0$.

To meet such a case, an extra factor $M_i$ is introduced into all tableau units. The diagonal elements $e_{11}, e_{22},\ldots, e_{ff}$ are defined in order

$$e_{11} = P_1 N_1,$$
$$e_{22} = (1-e_{11})P_2 N_2,$$

and generally
$$e_{ii} = M_i P_i N_i,$$

where
$$M_i = 1-e_{11}-e_{22}-\ldots-e_{i-1,i-1}.$$

Since $N_j P_i = 0$ for $i < j$, clearly

$$e_{jj}e_{ii} = 0 \quad (i < j),$$

and
$$e_{ii} M_i = e_{ii}.$$

Thus
$$\begin{aligned} e_{ii}^2 &= M_i P_i N_i P_i N_i \\ &= M_i P_i N_i \\ &= e_{ii}. \end{aligned}$$

Lastly,

$$\begin{aligned} e_{ii}e_{jj} &= e_{ii}(1-e_{11}-e_{22}-\ldots-e_{ii}-\ldots-e_{j-1,j-1})P_jN_j \\ &= 0 \quad (i<j). \end{aligned}$$

The exact form of $M_j$ is of importance. The factor $M_j$ may be taken as unity unless there is a tableau $A_i$ preceding $A_j$ such that the set of symbols in each row of $A_j$ appear in different columns of $A_i$. If there is such a tableau, then, clearly, a rearrangement of each column of $A_i$ will bring the symbols which are in the first row of $A_j$ into the first row of $A_i$; and similarly for every subsequent row. That is, there is a substitution $S_i$ belonging to $N_i$ which will transform $A_i$ into a tableau (not standard) in which the symbols in each row, except for order, are the same as the symbols in the corresponding row of $A_j$. A further permutation $T_j$, belonging to $P_j$, will transform this tableau into $A_j$. Hence

$$\sigma_{ij} = S_i T_j,$$

where $S_i$ belongs to $N_i$ and $T_j$ to $P_j$.

Clearly $$T_j P_j = P_j,$$

and $$N_i S_i = \pm N_i,$$

the sign being + or − according as $S_i$ is a positive or a negative permutation.

Let $\theta_{ij} = +1$ or $-1$ according as $S_i$ is a positive or a negative permutation.

Hence $$N_i\sigma_{ij}P_j = \theta_{ij}N_iP_j.$$

If, corresponding to $A_j$, there is only one tableau $A_i$ such that $N_iP_j \neq 0$, and further $M_i = 1$, we shall have

$$\begin{aligned} M_jP_jN_j &= (1-P_iN_i)P_jN_j \\ &= P_jN_j-P_iN_iP_jN_j \\ &= P_jN_j-\theta_{ij}P_iN_i\sigma_{ij}P_jN_j \\ &= P_jN_j-\theta_{ij}\sigma_{ij}(P_jN_j)^2 \\ &= (1-\theta_{ij}\sigma_{ij})P_jN_j. \end{aligned}$$

Thus we may take $$M_j = 1-\theta_{ij}\sigma_{ij}.$$

If there are several such tableaux $A_i$, assuming first that $M_i = 1$ in each case, $$M_j = 1-\sum_i \theta_{ij}\sigma_{ij}.$$

Allowing for the factors $M_i$, when these are not unity, we see that in the general case

$$M_j = 1 - \sum_i \theta_{ij}\sigma_{ij} + \sum_i\sum_k \theta_{ki}\theta_{ij}\sigma_{kj} - \sum_i\sum_k\sum_l \theta_{lk}\theta_{ki}\theta_{ij}\sigma_{lj} + \dots, \tag{5.4;5}$$

the summation being first with respect to every tableau $A_i$ such that $N_i P_j \neq 0$, and for each $i$, with respect to every tableau $A_k$ such that $N_k P_i \neq 0$, and so on. Since $\dots l < k < i < j$, the series terminates.

VI. *If the $f^{(\lambda)}$ standard tableaux $A_1, A_2, \dots, A_f$ are in the assigned order, and if $P_i$ is $f^{(\lambda)}/h$ times the product of the symmetric groups on the symbols in the rows of $A_i$, $N_i$ is the product of the negative symmetric groups on the symbols in the columns of $A_i$, $\sigma_{ij}$ is the permutation which transforms $A_i$ into $A_j$, and $M_i$ is defined by equation* (5.4; 5), *then the equations*

$$e_{ij} = M_i P_i \sigma_{ij} N_j \tag{5.4;6}$$

*give a simple matrix representation of the sub-algebra.*

To find the matrix in the sub-algebra which corresponds to a given permutation $S$, it would be necessary to solve $h$ equations. Fortunately these can be solved for the general case as follows.

Let $$e_{ij} = \sum a^{ij}_k S_k^{-1},$$
and let $$S_k = \sum b^{ij}_k e_{ij} + \phi_k,$$
where $\phi_k$ belongs to other sub-algebras.

Then the trace of $e_{ij} S_k$

$$\begin{aligned} &= \text{trace of } \sum a^{ij}_q S_q^{-1} S_k \\ &= h a^{ij}_k \\ &= \text{trace of } e_{ij} \sum b^{qr}_k e_{qr} \\ &= f^{(\lambda)} b^{ji}_k. \end{aligned}$$

Hence $$b^{ij}_k = h a^{ji}_k / f^{(\lambda)}.$$

VII. *The matrix corresponding to the group element $S_i$ in the sub-algebra of the Frobenius algebra corresponding to the partition* ($\lambda$) *is* $[b^{st}_k]$, *where $b^{st}_k$ is the coefficient of $S_k^{-1}$ in the expression*

$$h M_t P_t \sigma_{ts} N_s / f^{(\lambda)}.$$

As an example we consider the sub-algebra corresponding to the partition ($2^2$) for the symmetric group on the four symbols $\alpha$, $\beta$, $\gamma$, $\delta$. There are two standard tableaux

$$A_1 = \begin{pmatrix} \alpha, & \beta \\ \gamma, & \delta \end{pmatrix}, \qquad A_2 = \begin{pmatrix} \alpha, & \gamma \\ \beta, & \delta \end{pmatrix}.$$

Hence we have

$$\begin{aligned} 12P_1 &= I+(\alpha\beta)+(\gamma\delta)+(\alpha\beta)(\gamma\delta), \\ 12P_2 &= I+(\alpha\gamma)+(\beta\delta)+(\alpha\gamma)(\beta\delta), \\ N_1 &= I-(\alpha\gamma)-(\beta\delta)+(\alpha\gamma)(\beta\delta), \\ N_2 &= I-(\alpha\beta)-(\gamma\delta)+(\alpha\beta)(\gamma\delta). \end{aligned}$$

We may take $M_1 = M_2 = 1$.

Theorem VI gives

$$\begin{aligned} 12e_{11} = {} & I+(\alpha\beta)+(\gamma\delta)+(\alpha\beta)(\gamma\delta)-(\alpha\gamma)-(\alpha\beta\gamma)-(\alpha\gamma\delta)- \\ & -(\alpha\beta\gamma\delta)-(\beta\delta)-(\alpha\delta\beta)-(\beta\delta\gamma)-(\alpha\delta\gamma\beta)+(\alpha\gamma)(\beta\delta)+(\alpha\delta\beta\gamma)+ \\ & +(\alpha\gamma\beta\delta)+(\alpha\delta)(\beta\gamma), \\ 12e_{22} = {} & I+(\alpha\gamma)+(\beta\delta)+(\alpha\gamma)(\beta\delta)-(\alpha\beta)-(\alpha\gamma\beta)-(\alpha\beta\delta)-(\alpha\gamma\beta\delta)- \\ & -(\gamma\delta)-(\alpha\delta\gamma)-(\beta\gamma\delta)-(\alpha\delta\beta\gamma)+(\alpha\beta)(\gamma\delta)+(\alpha\delta\gamma\beta)+(\alpha\beta\gamma\delta)+(\alpha\delta)(\beta\gamma), \\ 12e_{12} = {} & (\beta\gamma)+(\alpha\gamma\beta)+(\beta\gamma\delta)+(\alpha\gamma\delta\beta)-(\alpha\beta\gamma)-(\alpha\gamma)-(\alpha\beta\gamma\delta)- \\ & -(\alpha\gamma\delta)-(\beta\delta\gamma)-(\alpha\delta\gamma\beta)-(\beta\delta)-(\alpha\delta\beta)+(\alpha\beta\delta\gamma)+(\alpha\delta\gamma)+(\alpha\beta\delta)+(\alpha\delta), \\ 12e_{21} = {} & (\beta\gamma)+(\alpha\beta\gamma)+(\beta\delta\gamma)+(\alpha\delta\beta\gamma)-(\alpha\gamma\beta)-(\alpha\beta)-(\alpha\gamma\beta\delta)- \\ & -(\alpha\beta\delta)-(\beta\gamma\delta)-(\alpha\beta\delta\gamma)-(\gamma\delta)-(\alpha\delta\gamma)+(\alpha\gamma\delta\beta)+(\alpha\delta\beta)+(\alpha\gamma\delta)+(\alpha\delta). \end{aligned}$$

From Theorem VII, picking out coefficients, we obtain

$I$, $(\alpha\beta)(\gamma\delta)$, $(\alpha\delta)(\beta\gamma)$, $(\alpha\gamma)(\beta\delta)$ are represented by $\begin{bmatrix} 1, & \\ & 1 \end{bmatrix}$,

$(\alpha\beta)$, $(\gamma\delta)$, $(\alpha\delta\beta\gamma)$, $(\alpha\gamma\beta\delta)$ $\begin{bmatrix} 1, & -1 \\ & -1 \end{bmatrix}$,

$(\alpha\gamma)$, $(\beta\delta)$, $(\alpha\beta\gamma\delta)$, $(\alpha\delta\gamma\beta)$ $\begin{bmatrix} -1, & \\ -1, & 1 \end{bmatrix}$,

$(\alpha\beta\gamma)$, $(\alpha\gamma\delta)$, $(\alpha\delta\beta)$, $(\beta\delta\gamma)$ $\begin{bmatrix} & -1 \\ 1, & -1 \end{bmatrix}$,

$(\beta\gamma)$, $(\alpha\delta)$, $(\alpha\beta\delta\gamma)$, $(\alpha\gamma\delta\beta)$ $\begin{bmatrix} & 1 \\ 1, & \end{bmatrix}$,

$(\alpha\gamma\beta)$, $(\alpha\beta\delta)$, $(\beta\gamma\delta)$, $(\alpha\delta\gamma)$ $\begin{bmatrix} -1, & 1 \\ -1, & \end{bmatrix}$.

### Young tableaux with repeated symbols

If a substitutional expression is multiplied both on the left and on the right by a factor $[I+(\alpha\beta)]$, the result will be symmetric with respect to these two symbols $\alpha$ and $\beta$, i.e. the symbols may be interchanged without altering the result. This applies especially to the characteristic units defined by Young tableaux. Two Young tableaux which are identical save that the symbols $\alpha$ and $\beta$ are interchanged will lead to the same resulting expression. It is sometimes convenient in such circumstances to write the tableaux with identical symbols, say two $\alpha$'s in the positions of the $\alpha$ and $\beta$. We say that the symbol $\alpha$ is repeated in the Young tableau.

The writing of a tableau thus with a repeated symbol is purely conventional, and the identical symbols must not be treated as the *same* symbol, e.g. from the tableau

$$\begin{pmatrix} \alpha, & \beta \\ \beta & \end{pmatrix}$$

to obtain the corresponding substitutional expression the $\beta$'s must be formally distinguished by suffixes or otherwise. It would be incorrect to obtain the product $$[I+(\alpha\beta)][I-(\alpha\beta)] = 0.$$

Similarly, if the substitutional expression is to be multiplied on the left and on the right by the symmetric group on $r$ symbols $\alpha_1, \alpha_2, \ldots, \alpha_r$, these symbols in the Young tableau may be replaced by the symbol $\alpha$ repeated $r$ times, since interchanges amongst these symbols will not affect the result.

One result is immediately obvious.

*If a symbol is repeated in the same column of a Young tableau, the resulting substitutional expression is zero.*

It is convenient to arrange the order of the symbols used to decide which tableaux are standard, so that the repeated symbols are consecutive. We can then refer to *standard Young tableaux with repeated symbols.*

If two tableaux $A_i$ and $A_j$ become identical when the given symbols are made identical, then clearly $P_i N_i$ and $P_j N_j$ become equal. The factors $M_i$ and $M_j$ may, however, be very different, but inspection shows that $M_i P_i N_i$ and $M_j P_j N_j$ must become equal also. For suppose that $N_k P_i$ or $N_k P_j \neq 0$. Then $k < i$ and $k < j$, and since $P_i N_i$ and $P_j N_j$ become equal, then $P_k N_k P_i N_i$ and $P_k N_k P_j N_j$ become equal also. It follows that $M_i P_i N_i$ and $M_j P_j N_j$ become equal.

It will be shown later that the matrix of a given sub-algebra, which corresponds to the direct product of symmetric groups on $p_1, p_2, \ldots, p_k$ symbols respectively, has rank equal to the number of non-zero standard Young tableaux that can be built with $k$ symbols which are repeated $p_1, p_2, \ldots, p_k$ times respectively.

# VI

# IMMANANTS AND *S*-FUNCTIONS

## 6.1. Immanants of a matrix (1)

LET $[a_{st}]$ be a matrix of order $n^2$. Let $S$ denote any permutation $e_1, e_2,\ldots, e_n$ of the numbers $1, 2,\ldots, n$. We denote by $P_S$ the product

$$P_S = a_{1e_1}a_{2e_2}\ldots a_{ne_n}.$$

There are clearly $n!$ products $P_S$ corresponding to the $n!$ permutations of the symmetric group.

Let $\chi^{(\lambda)}$ be the character of the symmetric group of order $n!$ corresponding to the partition $(\lambda) \equiv (\lambda_1, \lambda_2,\ldots, \lambda_p)$ of $n$.

*The immanant of the matrix* $[a_{st}]$ *corresponding to the partition* $(\lambda)$ *is denoted by* $|a_{st}|^{(\lambda)}$ *and is defined by the equation*

$$|a_{st}|^{(\lambda)} = \sum \chi^{(\lambda)}(S).P_S,$$

*the summation being with respect to the* $n!$ *permutations of the symmetric group.*

Since the character $\chi^{(n)}$ is unity for every operation of the group, and the character $\chi^{(1^n)}$ is $+1$ for a positive permutation and $-1$ for a negative permutation, the permanent and the determinant of a matrix are special cases of immanants.

$$\overset{+}{|}a_{st}\overset{+}{|} = |a_{st}|^{(n)},$$

$$|a_{st}| = |a_{st}|^{(1^n)}.$$

The matrix $[a_{st}]$ of order 9 has thus three immanants, the permanent, the determinant, and

$$|a_{st}|^{(21)} = 2a_{11}a_{22}a_{33} - a_{12}a_{23}a_{31} - a_{13}a_{21}a_{32}.$$

Matrices of orders $4^2$, $5^2$, and $6^2$ have respectively 5, 7, and 11 immanants.

It is well known that the determinant of a matrix has the remarkable property that it is equal to the determinant of any transform of the matrix. The same is not true in general of any other immanant, but every immanant has the following property:

I. *If $A$ is any matrix and $B$ is a permutation matrix of the same order, then*

$$|A|^{(\lambda)} = |B^{-1}AB|^{(\lambda)}.$$

If $B$ is the permutation matrix corresponding to the permutation

$T$, then the effect of transforming $A$ by $B$ is to replace each product $P_S$ by the product $P_{T^{-1}ST}$. Since

$$\chi^{(\lambda)}(S) = \chi^{(\lambda)}(T^{-1}ST)$$

the immanant is unchanged.

## 6.2. Schur functions

We consider now an application of the preceding theory to the theory of symmetric functions.

Consider the symmetric functions of $n$ quantities

$$\alpha_1,\ \alpha_2,\ \alpha_3,\ \ldots,\ \alpha_n.$$

These symmetric functions are usually associated with the equation whose roots are $\alpha_1$, $\alpha_2,\ldots,$ $\alpha_n$, namely

$$g(x) \equiv \prod (x-\alpha_r) \equiv x^n - a_1 x^{n-1} + a_2 x^{n-2} - \ldots + (-1)^n a_n = 0.$$

By actual multiplication it is clear that the coefficient $a_r$ of $(-1)^r x^{n-r}$ is the sum of the $\binom{n}{r}$ products of $r$ different quantities $\alpha_i$.

It is more convenient to associate the symmetric functions with the equation whose roots are the reciprocals of the quantities $\alpha_i$, namely

$$f(x) \equiv \prod (1-\alpha_r x) = 1 - a_1 x + a_2 x^2 - \ldots + (-1)^n a_n x^n = 0. \qquad (6.2;1)$$

It is then possible to take a limiting case as the number of roots becomes infinite.

By formal division $1/f(x)$ may be expanded in a series of ascending powers of $x$,

$$\begin{aligned} F(x) &\equiv 1/f(x) \equiv 1/\prod (1-\alpha_r x) \\ &\equiv \prod (1+\alpha_r x+\alpha_r^2 x^2+\alpha_r^3 x^3+\ldots) \\ &\equiv 1+h_1 x+h_2 x^2+\ldots+h_r x^r+\ldots . \end{aligned} \qquad (6.2;2)$$

Hence, by comparing coefficients, $h_r$ is the sum of the homogeneous products of degree $r$ of the quantities $\alpha_1$, $\alpha_2,\ldots,$ $\alpha_n$.

The quantities $a_r$ and $h_r$ are clearly symmetric functions, and are called respectively the elementary symmetric functions and the homogeneous product sums of $\alpha_1$, $\alpha_2,\ldots,$ $\alpha_n$. There is a third type of symmetric function of some importance, namely

$$S_r = \sum_{i=1}^{n} \alpha_i^r.$$

Part of the theory of symmetric functions concerns the formulae which connect these three types of symmetric function.

Since

$$1 \equiv f(x)F(x) \equiv (1-a_1x+a_2x^2-\ldots)(1+h_1x+h_2x^2+\ldots),$$

we have, by comparing coefficients,

$$a_r-a_{r-1}h_1+a_{r-2}h_2-\ldots+(-1)^r h_r = 0. \qquad (6.2;3)$$

Again, $\log f(x) = \sum \log(1-\alpha_r x)$.

Hence, the dash denoting differentiation,

$$\begin{aligned} f'(x)/f(x) &= \sum -\alpha_r/(1-\alpha_r x) \\ &= -\sum (\alpha_r+\alpha_r^2 x+\alpha_r^3 x^2+\ldots) \\ &= -\sum S_r x^{r-1}, \\ f'(x) &= -f(x)\sum S_r x^{r-1}, \end{aligned}$$

$$\begin{aligned} a_1-2a_2x+3a_3x^2-4a_4x^4+\ldots \\ = (1-a_1x+a_2x^2-\ldots)(S_1+S_2x+S_3x^2+\ldots). \end{aligned}$$

By comparison of coefficients we obtain

$$ra_r = S_1a_{r-1}-S_2a_{r-2}+S_3a_{r-3}-\ldots\pm S_n. \qquad (6.2;4)$$

These relations were first given by Newton:

$$\left.\begin{aligned} S_1-a_1 &= 0, \\ S_2-S_1a_1+2a_2 &= 0, \\ S_3-S_2a_1+S_1a_2-3a_3 &= 0, \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \quad\cdot \\ S_r-S_{r-1}a_1+\ldots+(-1)^r ra_r &= 0. \end{aligned}\right\} \qquad (6.2;5)$$

Solving these equations for the $a_r$'s in terms of the $S_r$'s by means of determinants, we obtain

$$r!\,a_r = |Z_r|, \qquad (6.2;6)$$

where

$$[Z_r] = \begin{bmatrix} S_1, & 1 & & & \\ S_2, & S_1, & 2 & & \\ S_3, & S_2, & S_1, & 3 & \\ & & & & \\ \cdot & \cdot & \cdot & \cdot & r-1 \\ S_r, & S_{r-1}, & S_{r-2}, & \ldots, & S_1 \end{bmatrix} \qquad (6.2;7)$$

In a similar manner

$$\begin{aligned} F'(x)/F(x) &= \sum S_r x^{r-1}, \\ F'(x) &= F(x)\sum S_r x^{r-1} \end{aligned}$$

$$h_1+2h_2x+3h_3x^2+\ldots = (1+h_1x+h_2x^2+\ldots)(S_1+S_2x+S_3x^2+\ldots),$$

$$rh_r = S_1h_{r-1}+S_2h_{r-2}+\ldots+S_r. \qquad (6.2;8)$$

These relations are due to Brioschi:

$$\left.\begin{aligned} S_1-h_1 &= 0, \\ S_2+S_1h_1-2h_2 &= 0, \\ S_3+S_2h_1+S_1h_2-3h_3 &= 0, \\ & \\ S_r+S_{r-1}h_1+\ldots-rh_r &= 0. \end{aligned}\right\} \tag{6.2; 9}$$

Hence

$$r!\,h_r = \begin{vmatrix} S_1, & -1 & & & \\ S_2, & S_1, & -2 & & \\ S_3, & S_2, & S_1, & -3 & \\ & & & & \\ S_r, & S_{r-1}, & & \ldots, & S_1 \end{vmatrix}$$

Now every non-zero product $P_S$ as defined in § 6.1, obtained from the matrix $[Z_r]$ corresponding to a positive permutation $S$, must contain an even number of terms from the diagonal above the leading diagonal, and corresponding to a negative permutation, an odd number. Hence the effect of changing the sign of every term in this diagonal is to change the sign of every product $P_S$ corresponding to a negative permutation $S$, and thus to interchange the permanent and the determinant of the matrix.

Hence $$r!\,h_r = \overset{+}{|}Z_r\overset{+}{|}. \tag{6.2; 10}$$

Equations (6.2; 6) and (6.2; 10) suggest that other immanants of the matrix $[Z_r]$ besides the permanent and the determinant may be of interest. We therefore define the *Schur functions*, or, as we shall call them, $S$-functions (2), from the immanants of the matrix $[Z_r]$.

DEFINITION. *If* $(\lambda) \equiv (\lambda_1, \lambda_2,\ldots, \lambda_p)$ *is a partition of* $r$ *with the parts in descending order, the* $S$*-function* $\{\lambda\} \equiv \{\lambda_1, \lambda_2,\ldots, \lambda_p\}$ *is defined by the equation*
$$r!\{\lambda\} = |Z_r|^{(\lambda)}. \tag{6.2; 11}$$

Equations (6.2; 6) and (6.2; 10) give

$$\left.\begin{aligned} \{r\} &= h_r, \\ \{1^r\} &= a_r. \end{aligned}\right\} \tag{6.2; 12}$$

Let $\rho$ be a class of order $h_\rho$ of the symmetric group of order $r!$. We find an expression for the sum of the products $\sum\limits_{\rho} P_S$ obtained from the matrix $[Z_r]$ corresponding to the $h_\rho$ permutations of the class $\rho$.

Firstly, let $\rho = (p_1, p_2, ..., p_i)$, and let the orders $p_1, ..., p_i$ of the cycles be all different. One permutation of this class is obtained by cyclically permuting the first $p_1$ symbols, then the following $p_2$ symbols, and so on. The corresponding product is

$$\begin{aligned} P_S &= 1.2.3...(p_1-1)S_{p_1}(p_1+1)...(p_1+p_2-1)S_{p_2}(p_1+p_2+1)... \\ &= r!\, S_{p_1} S_{p_2} ... S_{p_i} / p_1(p_1+p_2)...(p_1+p_2+...+p_i). \end{aligned}$$

The only other permutations of this class which give non-zero products $P_S$ are obtained in a like manner, but by permuting the orders of the cycles, $p_1, ..., p_i$.

Hence $$\sum_\rho P_S = k_\rho S_{p_1} S_{p_2} ... S_{p_i},$$

where $$k_\rho = r! \sum 1/p_1(p_1+p_2)...(p_1+p_2+...+p_i)$$

summed for the $i!$ permutations of the suffixes.

It can be shown by induction that

$$\sum 1/p_1(p_1+p_2)...(p_1+p_2+...+p_i) = 1/p_1 p_2 ... p_i.$$

Assuming it true for $i$, we have, summing for the $(i+1)$ combinations of the $(i+1)$ suffixes $i$ at a time,

$$\begin{aligned} &\sum 1/p_1(p_1+p_2)...(p_1+p_2+...+p_{i+1}) \\ &\frac{1}{p_1 p_2 ... p_{i+1}}\left[\frac{p_1}{p_1+p_2+...+p_{i+1}} + \frac{p_2}{p_1+p_2+...+p_{i+1}} + \right. \\ &\qquad \left. + ... + \frac{p_{i+1}}{p_1+p_2+...+p_{i+1}}\right] = 1/p_1 p_2 ... p_{i+1}. \end{aligned}$$

Hence $$k_\rho = r!/p_1 p_2 ... p_i.$$

If, however, $\rho$ contains cycles of equal order, let it be the class $(1^a\, 2^b\, 3^c ...)$, where $a+b+c+... = i$. All the $i!$ permutations of the cycles are not now distinct, but consist of the $i!/a!\,b!\,c!...$ distinct permutations repeated $a!\,b!\,c!...$ times.

Thus the value $$r!/p_1 p_2 ... p_i = r!/1^a 2^b 3^c ...$$

is $a!\,b!\,c!...$ times $k_\rho$, and

$$\begin{aligned} k_\rho &= r!/1^a a!\, 2^b b!\, 3^c c! ... \\ &= h_\rho, \end{aligned}$$

the order of the class $\rho$, from (3.6; 1).

Thus in every case
$$\sum P_S = h_\rho S_\rho, \tag{6.2; 13}$$
where $S_\rho$ is as defined by equation (5.2; 3), and we arrive at the very important formula
$$r!\{\lambda\} = \sum h_\rho \chi_\rho^{(\lambda)} S_\rho. \tag{6.2; 14}$$

The expression of the $S$-functions in terms of the symmetric functions $S_i$ may now be read from the tables of characters of the symmetric groups.

Conversely, the products of the $S_r$'s may be expressed in terms of $S$-functions as follows. From (6.2; 14) we obtain
$$\begin{aligned}\sum_\lambda \chi_\rho^{(\lambda)}\{\lambda\} &= \sum h_{\rho'} \chi_\rho^{(\lambda)} \chi_{\rho'}^{(\lambda)} S_{\rho'}/r! \\ &= S_\rho,\end{aligned}$$
so that
$$S_\rho = \sum_\lambda \chi_\rho^{(\lambda)}\{\lambda\}. \tag{6.2; 15}$$

Thus from the table of characters of the symmetric group of order 3!

| Class . . | $(1^3)$ | $(2\,1)$ | $(3)$ |
|---|---|---|---|
| Order. . | 1 | 3 | 2 |
| $\chi^{(3)}$ | 1 | 1 | 1 |
| $\chi^{(21)}$ | 2 | 0 | $-1$ |
| $\chi^{(1^3)}$ | 1 | $-1$ | 1 |

we obtain
$$\begin{aligned}6h_3 = 6\{3\} &= S_1^3 + 3S_1S_2 + 2S_3, \\ 6\{2\,1\} &= 2S_1^3 - 2S_3, \\ 6a_3 = 6\{1^3\} &= S_1^3 - 3S_1S_2 + 2S_3, \\ S_1^3 &= \{3\} + 2\{2\,1\} + \{1^3\}, \\ S_1S_2 &= \{3\} - \{1^3\}, \\ S_3 &= \{3\} - \{2\,1\} + \{1^3\}.\end{aligned}$$

It follows that if the number $n$ of the quantities $\alpha_1$ from which the symmetric functions are formed is greater than $r$, then there are exactly sufficient $S$-functions corresponding to partitions of $r$ to express the general symmetric function of weight $r$ without ambiguity. If, however, $n < r$, the quantities $S_i$ are algebraically dependent, and thus the products $S_\rho$, and hence the $S$-functions of weight $r$, are linearly dependent. We shall show that this linear dependency manifests itself in a remarkably simple manner, the $S$-functions corresponding to partitions into more than $n$ parts being identically equal to zero.

## 6.3. Properties of $S$-functions

**Expressions as quotient of determinants and as determinants**

From Frobenius's formula (5.2; 8)

$$S_\rho \Delta(\alpha_1,\ldots,\alpha_n) = \sum \pm \chi_\rho^{(\lambda)} \alpha_1^{\lambda_1+n-1} \alpha_2^{\lambda_2+n-2} \ldots \alpha_n^{\lambda_n}$$

we obtain

$$\sum h_\rho \chi_\rho^{(\lambda)} S_\rho \Delta(\alpha_1,\ldots,\alpha_n) = \sum_{\rho\mu} \pm h_\rho \chi_\rho^{(\lambda)} \chi_\rho^{(\mu)} \alpha_1^{\mu_1+n-1} \alpha_2^{\mu_2+n-2} \ldots \alpha_n^{\mu_n}$$
$$= r! \sum \pm \prod_i \alpha_i^{\lambda_i+n-i}.$$

Hence $$\{\lambda\} = \frac{1}{r!} \sum h_\rho \chi_\rho^{(\lambda)} S_\rho = \frac{\sum \pm \prod_i \alpha_i^{\lambda_i+n-i}}{\sum \pm \prod_i \alpha_i^{n-i}}$$

In each case the summation is taken with respect to all permutations of the suffixes, the negative sign being taken for a negative permutation.

Hence $$\{\lambda\} = \frac{|\alpha_s^{\lambda_t+n-t}|}{|\alpha_s^{n-t}|}. \tag{6.3; 1}$$

The numerator and the denominator of this quotient are alternating functions of the $\alpha$'s, or *alternants*; i.e. they are unaltered by a positive permutation of the $\alpha$'s, but are changed in sign by a negative permutation. A quotient of two alternants must, of course, be a symmetric function. From such a definition $S$-functions have been studied by Jacobi, Trudi, Naegelsbach, and Kostka long before group characters were discovered by Frobenius, and Muir refers to these functions under the name *bi-alternants* (**3**). Jacobi, and independently Trudi, expressed this quotient as a determinant in which the elements are the symmetric functions $h_r$. Naegelsbach expressed the same quotient as a determinant in which the elements are the symmetric functions $a_r$. Later Kostka proved Trudi's and Naegelsbach's theorems for himself in a more elegant manner, and obtained further properties of the functions, constructing tables connecting the functions with other symmetric functions. Schur (**4**) was the first to define the functions with any reference to group characters, and hence they are named after him.

We obtain first the Jacobi-Trudi equation. Denote by $(a,b,c,\ldots)_r$

the symmetric function $h_r$ of the quantities $\alpha_a$, $\alpha_b$, $\alpha_c,\dots$, and denote $(1,2,3,\dots,q)_r$ by $h_r^{(q)}$, so that $h_r^{(n)} = h_r$. Then it is easily verified that

$$(a,b,\dots,g,k)_r-(a,b,\dots,g,k')_r = (\alpha_k-\alpha_{k'})(a,b,\dots,g,k,k')_{r-1},$$

and also

$$h_q^{(p+1)} = h_q^{(p)}+\alpha_{p+1}h_{q-1}^{(p+1)}.$$

In the determinant $|\alpha_s^{\lambda_t+n-t}|$, subtract the first row from every subsequent row, and remove the factor $(\alpha_s-\alpha_1)$ from the $s$th row. The element in the $s$th row, $t$th column becomes, for $s > 1$, $(1,s)_{\lambda_t+n-t-1}$.

Subtract the second row from every subsequent row and remove the factor $(\alpha_s-\alpha_2)$ from the $s$th row ($s > 2$). Since

$$(1,s)_r-(1,2)_r = (\alpha_s-\alpha_2)(1,2,s)_{r-1},$$

the element in the $s$th row, $t$th column, for $s > 2$, becomes

$$(1,\ 2,\ s)_{\lambda_t+n-t-2}.$$

Continuing thus we obtain

$$|\alpha_s^{\lambda_t+n-t}| = \prod_{i<j}(\alpha_j-\alpha_i)\,|h^{(s)}_{\lambda_t+n-t-s+1}|.$$

Now consecutively, for $s = 2,3,4$, etc., add $\alpha_s$ times the $s$th row to the $(s-1)$th row. $h^{(s)}_{\lambda_t+n-t-s+1}$ is replaced by $h^{(s+1)}_{\lambda_t+n-t-s+1}$. Similarly, add $\alpha_{s+1}$ times the $s$th row to the $(s-1)$th row, and continue thus The upper suffix of each term is thus raised up to $n$, so that

$$|\alpha_s^{\lambda_t+n-t}| = \prod_{i<j}(\alpha_j-\alpha_i)\,|h_{\lambda_t+n-t-s+1}|.$$

By reversing the order of the rows the right-hand side becomes

$$\prod_{i<j}(\alpha_i-\alpha_j)\,|h_{\lambda_t+s-t}|.$$

It is convenient in this determinant, to interchange the rows with the columns. Hence

$$|\alpha_s^{\lambda_t+n-t}| = \prod_{i<j}(\alpha_i-\alpha_j)\,|h_{\lambda_s-s+t}|,$$

and we obtain

$$\{\lambda\} = \frac{|\alpha_s^{\lambda_t+n-t}|}{|\alpha_s^{n-t}|} = |h_{\lambda_s-s+t}|. \qquad (6.3;2)$$

The second equation of this is the Jacobi-Trudi equation.

Now suppose $n \geqslant r$. If $x$ is any symmetric function of weight $r$, it can be expressed without ambiguity in terms of the functions $S_1$, $S_2,\dots$, $S_r$. In this expression replace $S_2$, $S_4$, $S_6,\dots$ respectively by $-S_2$, $-S_4$, $-S_6,\dots$. The result is another symmetric function of weight $r$ which we denote by $\tilde{x}$. Clearly, if $y = \tilde{x}$, then $x = \tilde{y}$. Then from (6.2; 14) and Theorems IV and V, § 5.3,

$$\tilde{h}_r = a_r, \qquad \tilde{a}_r = h_r,$$

and more generally if $\mu \equiv (\mu_1, \mu_2,\dots, \mu_q)$ is the partition of $r$ conjugate to $(\lambda)$,

$$\widetilde{\{\mu\}} = \{\lambda\}.$$

Hence

$$\begin{aligned}\{\lambda\} &= \widetilde{\{\mu\}} \\ &= \widetilde{|h_{\mu_s-s+t}|} \\ &= |a_{\mu_s-s+t}|.\end{aligned}$$

If $(\mu)$ is the partition conjugate to $(\lambda)$, then

$$\{\lambda\} = |a_{\mu_s-s+t}|. \qquad (6.3;\,3)$$

The proof is extended to the case $n < r$ by putting superfluous roots equal to zero.

This equation (6.3; 3), or rather the equation obtained on replacing $\{\lambda\}$ by the appropriate quotient of alternants, is equivalent to the equation of Naegelsbach and Kostka.

### Symmetric functions of the form $\sum \alpha_1^{a_1}\alpha_2^{a_2}\dots\alpha_p^{a_p}$ (5)

The above equations connect $S$-functions with alternating functions and with the symmetric functions $a_r$ and $h_r$. We now show how to express symmetric functions of the type $\sum \alpha_1^{a_1}\alpha_2^{a_2}\dots\alpha_p^{a_p}$ in terms of $S$-functions.

Since there are exactly sufficient $S$-functions of weight $r$ (for $n \geqslant r$) to express the general symmetric function of weight $r$, we can always express $\sum \alpha_1^{a_1}\dots\alpha_p^{a_p}$ in the form

$$\sum \alpha_1^{a_1}\dots\alpha_p^{a_p} = \sum K^{(\lambda)}\{\lambda\}.$$

By the following method we can evaluate the coefficients $K^{(\lambda)}$.

Since from (6.3; 1)

$$\{\lambda\}\Delta(\alpha_1,\dots, \alpha_n) = \sum \pm \alpha_1^{\lambda_1+n-1}\alpha_2^{\lambda_2+n-2}\dots\alpha_n^{\lambda_n},$$

we have

$$\sum \alpha_1^{a_1}\dots\alpha_p^{a_p}\Delta(\alpha_1,\dots, \alpha_n) = \sum \pm K^{(\lambda)}\alpha_1^{\lambda_1+n-1}\alpha_2^{\lambda_2+n-2}\dots\alpha_n^{\lambda_n}.$$

Hence

I. $$\sum \alpha_1^{a_1}\alpha_2^{a_2}\dots\alpha_p^{a_p} = \sum K^{(\lambda)}\{\lambda\},$$

*where* $K^{(\lambda)}$ *is the coefficient of* $\alpha_1^{\lambda_1+n-1}\alpha_2^{\lambda_2+n-2}\dots\alpha_n^{\lambda_n}$ *in the product*

$$\sum \alpha_1^{a_1}\alpha_2^{a_2}\dots\alpha_p^{a_p}\Delta(\alpha_1,\dots, \alpha_n).$$

Though the number of terms in $\Delta(\alpha_1,\dots, \alpha_n)$ may be very large, we can pick out the required coefficients very easily as the following example will show.

**Example.** To express $\sum \alpha_1^3 \alpha_2 \alpha_3$ in terms of *S*-functions. Here we must have $n \geqslant 5$. We take the simplest case $n = 5$. We have to pick out the coefficients in the product

$$(\sum \alpha_1^3 \alpha_2 \alpha_3)(\sum \pm \alpha_1^4 \alpha_2^3 \alpha_3^2 \alpha_4),$$

the second summation covering all permutations of the suffixes (including the omitted suffix 5), taking a negative sign for a negative permutation. In the product we need only consider terms with all indices different.

We set out the calculation first and follow with the explanation.

| 4 | 3 | 2 | 1 | 0 | | | |
|---|---|---|---|---|---|---|---|
| | | 3 | 1 | 1 | 4, 3, 5, 2, 1 | 1, 1, 1, 1, 1 | + |
| 1 | | | 3 | 1 | 5, 3, 2, 4, 1 | 1, 1, 1, 1, 1 | + |
| 1 | 1 | | | 3 | 5, 4, 2, 1, 3 | 1, 1, 1, 1, 1 | + |
| | 3 | 1 | 1 | | 4, 6, 3, 2, 0 | 2, 1, 1, 1 | — |
| 3 | 1 | 1 | | | 7, 4, 3, 1, 0 | 3, 1, 1 | + |
| 1 | 3 | 1 | | | 5, 6, 3, 1, 0 | 2, 2, 1 | — |

We set down in the first row of the first column the indices in the right-hand term of the product, namely 4, 3, 2, 1, 0. We have to add to these the indices 3, 1, 1 in any order so that the five indices resulting are all different. In the first instance we may add any of these to zero. In the second row we add 1 to zero. The sum is 1, and since this index must not be repeated, either the other 1, or the 3 must be added to 1. Following this principle we obtain the second and the third rows, and if 3 is added to zero, the fourth row. The other possibilities are given in the last three rows.

In the second column we give the corresponding sums of the indices, namely $\lambda_1+n-1$, $\lambda_2+n-2,\ldots$, $\lambda_n$ in some order. In the third column these are arranged in descending order and the numbers 4, 3, 2, 1, 0 respectively subtracted, giving the numbers $\lambda_1$, $\lambda_2,\ldots$, $\lambda_n$. In the fourth column a + or — sign is given according as the second column is a positive or a negative permutation of the descending order.

Hence $$\sum \alpha_1^3 \alpha_2 \alpha_3 = \{3\,1^2\}-\{2\,1^3\}-\{2^2 1\}+3\{1^5\}.$$

**Values of the *S*-functions of $\alpha_1,\ldots,\alpha_n$ when the weight $r > n$**

If the weight $r > n$, the products $S_\rho$ and hence the *S*-functions are not linearly independent. This dependency manifests itself in the simple manner indicated by the following theorem.

II. *An $S$-function of weight $r$ corresponding to a partition of $r$ into more than $n$ parts is identically zero.*

It is sufficient to prove it for the case when the number of parts exceeds the number of variables by one. The extension to the more general case is obtained by putting some of the variables $\alpha_i$ equal to zero.

Let $(\lambda) = (\lambda_1,\ldots,\lambda_p)$ be a partition of $r$ into $p = n+1$ parts. Consider the $S$-function $\{\lambda\}$ of $\alpha_1, \alpha_2,\ldots, \alpha_n, \alpha_{n+1}$, in which we shall put $\alpha_{n+1} = 0$.

We have
$$\{\lambda\} = \frac{|\alpha_s^{\lambda_t+n-t}|}{|\alpha_s^{n-t}|}.$$

The determinant in the numerator has in the $(n+1)$th row powers of $\alpha_{n+1}$ only, which are zero since $\alpha_{n+1} = 0$. But if the variables $\alpha_1, \alpha_2,\ldots, \alpha_n$ are all distinct and not zero, the denominator, which is the product of the differences, is not zero. Hence if $\alpha_{n+1} = 0$, then $\{\lambda\} = 0$. The equation $\{\lambda\} = 0$ is clearly an identity in the variables $\alpha_1, \alpha_2,\ldots, \alpha_n$, and remains true if some of these are equal, or are zero.

This proves the theorem.

### The multiplication of $S$-functions (6)

An account of $S$-functions would hardly be complete without a set of rules for multiplication. We set out the rules in the form of three theorems, namely III, IV, and V which follow.

III. *The product of two $S$-functions of weights $r$ and $s$ respectively is equal to the sum of integral multiples of $S$-functions of weight $r+s$.*

To prove the theorem we require the following lemma.

LEMMA. *An isomorphism exists between the multiplication of $S$-functions and the multiplication of corresponding characteristic units involving different sets of symbols.*

Consider the product of two $S$-functions $\{\lambda\}\{\mu\}$ of weights $r$ and $s$ respectively. Corresponding to $\{\lambda\}$ consider any characteristic unit of the symmetric group on the $r$ symbols
$$\beta_1, \beta_2, \ldots, \beta_r,$$
which corresponds to the simple character $\chi^{(\lambda)}$, and corresponding to $\{\mu\}$ a characteristic unit of the symmetric group on the $s$ symbols
$$\beta_{r+1}, \beta_{r+2}, \ldots, \beta_{r+s},$$
which corresponds to the simple character $\chi^{(\mu)}$.

The direct product of these groups is a subgroup of the symmetric group on the $r+s$ symbols. The product of two operations of these symmetric groups will have the sum of the cycles of the two operations. Hence if the operations belong to classes $\rho$ and $\rho'$ respectively, and the product to the class $\rho''$ of the symmetric group of order $(r+s)!$, then

$$S_\rho S_{\rho'} = S_{\rho''},$$

$S_\rho$ being defined by equation (5.2; 3).

Since from § 4.4, Theorem I, a characteristic unit corresponding to the simple character $\chi^{(\lambda)}$ has an aggregate of $h_\rho \chi_\rho^{(\lambda)}/r!$ operations from the class $\rho$, the proof of the lemma follows.

The proof of the theorem now follows immediately, for the product of two characteristic units, which are commutative, since they correspond to substitutions on different sets of symbols, must in the nature of the case be a characteristic unit.

We shall use as characteristic units those which correspond to standard Young tableaux corresponding to the given partition.

IV. *The S-functions obtained in the product $\{\lambda\}\times h_r$ are those which correspond to the Young tableaux that can be built by the addition of $r$ identical symbols to a tableau corresponding to the S-function $\{\lambda\}$, no two identical symbols appearing in the same column.*

We use $\alpha$ for the $r$ identical symbols, and for convenience number them $\alpha_1, \alpha_2,\ldots, \alpha_r$.

Let $A$ denote a Young tableau corresponding to the partition $(\lambda)$, and denote by $(A)$ the product of the symmetric groups of the rows times the product of the negative symmetric groups of the columns. $B$ is a tableau with one row containing the $r$ symbols $\alpha_1, \alpha_2,\ldots, \alpha_r$. The corresponding characteristic units are

$$\frac{f^{(\lambda)}}{n!}(A) \quad \text{and} \quad \frac{1}{r!}(B).$$

Two numbers of the Frobenius algebra are said to be equivalent if one can be transformed into the other. The symbol $\sim$ denotes equivalence.

The theorem is proved for this case if we can demonstrate that

$$\frac{\chi_0^{(\lambda)}}{n!}(A)\frac{1}{r!}(B) \sim \sum \frac{\chi_0^{(\nu)}}{(r+n)!}(C),$$

the summation on the right being taken over the tableaux $C$ defined in the theorem, $\chi^{(\nu)}$ being the corresponding character.

The modulus of the invariant sub-algebra corresponding to $\chi^{(\nu)}$ of the Frobenius algebra of the symmetric group of order $(n+r)!$ is equal to

$$\frac{\chi_0^{(\nu)}}{(n+r)!}[\Sigma\,(D)],$$

the summation being taken over the $\chi_0^{(\nu)}$ standard tableaux $D$, corresponding to $\chi^{(\nu)}$. The assigned order of the symbols needed to decide which tableaux are standard is any order of the $n$ symbols of $A$ that would make $A$ a standard tableau, followed by the $r$ symbols of $B$ taken in the same order.

Since the identical element of the group, and also the modulus of the sub-algebra, are unaltered by transformations, we may multiply both sides of the above equivalence by this modulus and equate coefficients of the identical element. The right-hand side becomes equal to

$$\frac{g_{\lambda r\nu}\,\chi_0^{(\nu)}}{(n+r)!},$$

where $g_{\lambda r\nu}$ is the number of tableaux $C$ which correspond to $\chi^{(\nu)}$. From the method of definition, each of the tableaux $C$ is standard.

Now suppose that the row of $r$ symbols in the tableau $B$ is divided in the tableau $C$ into $t$ rows containing respectively $a_1, a_2,\ldots, a_t$ of these symbols. Corresponding to one tableau $C$, there will be $(r!)/(a_1!\,a_2!\ldots a_t!)$ standard tableaux $D$ which can be obtained by the rearrangement of these symbols, the symbols in each row still being in the assigned order. The tableaux $D$ that do not correspond in this way to a tableau $C$ will make the product $(A)(B)(D)$ zero, from Theorem III.

Consider the coefficient of the identical element in the product $(A)(B)(D)$. The tableau $A$ contains only the first $n$ symbols, and the tableau $B$ only the last $r$ symbols. Hence the only significant terms from the tableau $D$ are those that contain no cycles involving both sets of symbols. Thus the required coefficient will be unaltered if we replace the tableau $D$ by that portion of it which contains the first $n$ symbols, which portion must be identical with $A$ if the product is not to be zero, and multiply by the symmetric groups on the $a_1$, $a_2,\ldots, a_t$ symbols from tableau $B$.

The coefficient of the identical element in the product

$$\frac{\chi_0^{(\lambda)}}{n!}(A)\frac{1}{r!}(B)\frac{\chi_0^{(\nu)}}{(n+r)!}(D)$$

must therefore be

$$\frac{\chi_0^{(\nu)}}{(n+r)!}\,\frac{1}{r!}\,a_1! \quad !\dots a_t!.$$

If we sum for the $(r!)/(a_1!\dots a_t!)$ tableaux $D$ which correspond to each tableau $C$, we obtain

$$\frac{\chi_0^{(\nu)}}{(n+r)!}$$

for each tableau $C$. The theorem follows.

V. *The S-functions appearing in the product* $\{\lambda_1,\dots,\lambda_p\}\{\mu_1,\dots,\mu_q\}$ *are those which correspond to the Young tableaux that can be built by adding to a Young tableau corresponding to* $\{\lambda\}$, $\mu_1$ *identical symbols* $\alpha$, $\mu_2$ *identical symbols* $\beta$, $\mu_3$ *identical symbols* $\gamma$, *etc., subject to two conditions:*

*Firstly, after the addition of each set of identical symbols we must have a regular Young tableau with no two identical symbols in the same column.*

*Secondly, if the total set of added symbols is read from right to left in the consecutive rows of the final tableau, we obtain a lattice permutation of* $\alpha^{\mu_1}\beta^{\mu_2}\gamma^{\mu_3}\dots$.

It follows from Theorem IV that the $S$-functions in the product

$$\{\lambda\}h_{\mu_1}h_{\mu_2}h_{\mu_3}\dots$$

correspond to all tableaux that can be built in accordance with Theorem V subject to the first condition only.

Again, taking $\{\lambda\}=\{0\}=1$, we have from Theorem III

$$h_{\mu_1}h_{\mu_2}h_{\mu_3}\dots=\{\mu\}+\sum\{\mu'\},$$

where $\{\mu'\}$ is summed for all $S$-functions, excepting only $\{\mu\}$, which correspond to standard Young tableaux that can be built with $\mu_1$ $\alpha$'s, $\mu_2$ $\beta$'s, $\mu_3$ $\gamma$'s, etc. As an example, from the tableaux

$$(\alpha\ \alpha\ \beta\ \gamma),\quad \begin{pmatrix}\alpha & \alpha & \beta\\ \gamma & & \end{pmatrix},\quad \begin{pmatrix}\alpha & \alpha & \gamma\\ \beta & & \end{pmatrix},$$

$$\begin{pmatrix}\alpha & \alpha\\ \beta & \gamma\end{pmatrix},\quad \begin{pmatrix}\alpha & \alpha\\ \beta & \\ \gamma & \end{pmatrix}$$

we deduce that

$$h_2h_1^2=\{2\,1^2\}+\{2^2\}+2\{3\,1\}+\{4\}.$$

Hence we have

$$\{\lambda\}\{\mu\} = \{\lambda\}h_{\mu_1}h_{\mu_2}h_{\mu_3}\ldots - \sum\{\lambda\}\{\mu'\}.$$

We will assume the theorem to be true for the products $\{\lambda\}\{\mu'\}$ and prove it for the product $\{\lambda\}\{\mu\}$. The principle of mathematical induction then allows us to deduce the truth for all cases, since the partitions of $n$ may be ordered so that for each partition $\{\mu\}$ the corresponding partitions $\{\mu'\}$ all precede $\{\mu\}$.

The proof is obtained by showing that for any tableau built according to the first condition of Theorem V which gives a non-lattice permutation of $\alpha^{\mu_1}\beta^{\mu_2}\gamma^{\mu_3}\ldots$, there corresponds a tableau built according to both conditions of Theorem V for a product $\{\lambda\}\{\mu'\}$, and conversely.

For a given non-lattice permutation of $\alpha^{\mu_1}\beta^{\mu_2}\gamma^{\mu_3}\ldots$, consider first the $\alpha$'s and the $\beta$'s only. Number the $\alpha$'s and $\beta$'s in the order of their appearance.

If $\beta_s$ precedes $\alpha_{t+1}$ and succeeds $\alpha_t$, it is said to be of index $s-t$, and is said to be of positive, zero, or negative index according as $s-t$ is positive, zero, or negative.

If the $\alpha$'s and $\beta$'s exhibit the lattice property, there is no $\beta$ of positive index.

Otherwise take the first $\beta$ of greatest (positive) index and replace it by an $\alpha$. This step is reversible, an essential part of the argument, for the proof depends upon an exact 1 : 1 correspondence. To reverse the step we renumber the $\alpha$'s and $\beta$'s, and take the last $\beta$ of greatest zero or positive index and replace the $\alpha$ immediately following it by a $\beta$, unless all $\beta$'s are of negative index, in which case we replace the first $\alpha$ in the permutation by a $\beta$.

Now take a Young tableau $M$ corresponding to $\{\mu\}$ with $\mu_1$ $\alpha$'s in the first row, $\mu_2$ $\beta$'s in the second row, etc. When a $\beta$ is converted into an $\alpha$ in our permutation, the last symbol in the second row of $M$ is moved up to the end of the first row.

The process is repeated both with the non-lattice permutation and with the tableau $M$ until there is no $\beta$ of positive index, i.e. as far as the $\alpha$'s and $\beta$'s are concerned the permutation exhibits the lattice property.

Next the $\beta$'s and $\gamma$'s only are considered, and each $\gamma$ is given an index relative to the $\beta$'s. If necessary the first $\gamma$ of greatest positive index is converted into a $\beta$, and at the same time a symbol is moved from row 3 to row 2 of $M$.

This step may destroy the lattice property of the $\alpha$'s and $\beta$'s. If so, the first $\beta$ of index $+1$, which may or may not be the symbol converted from a $\gamma$ to a $\beta$, is converted into an $\alpha$, and at the same time a symbol is moved from the second row to the first row of $M$.

The process is continued consecutively with the $\gamma$'s, $\delta$'s, etc., until we arrive at a lattice permutation of $\alpha^{\mu_1'}\beta^{\mu_2'}\gamma^{\mu_3'}\dots$, and the Young tableau $M$ now corresponds to the partition $(\mu')$.

To establish the 1 : 1 correspondence between the tableaux satisfying the first condition of Theorem V but yielding non-lattice permutations, and the tableaux obtained in the various multiplications $\{\lambda\}\{\mu'\}$, it is only necessary to show that the first condition of Theorem V remains satisfied in the transformed tableau.

Consider any two consecutive symbols in a given permutation of $\alpha^{\mu_1}\beta^{\mu_2}\gamma^{\mu_3}\dots$. If these two symbols are in natural order, e.g. $\alpha\beta$, $\alpha\gamma$, $\beta\gamma$, they are said to form a positive step; if in reversed order, a negative step; if they are the same symbol, a zero step. Then according to our method of transformation it will be seen that a negative or zero step never becomes positive, while a positive step always remains positive. This is sufficient to ensure that the first condition of Theorem V is always satisfied, and the proof of the theorem is complete.

Example. To form the product of

$$\{4\,3\,1\}\times\{2^2\,1\}$$

we add to the tableau corresponding to $\{4\ 3\ 1\}$ the symbols $\alpha^2\beta^2\gamma$. For convenience we replace the symbols of the tableau $\{4\ 3\ 1\}$ by 0's. The corresponding tableaux are

$$\begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&\beta&\beta&\\ 0&\gamma&&&& \end{array}\qquad \begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&\beta&\beta&\\ 0&&&&& \end{array}\qquad \begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&\beta&&\\ 0&\beta&\gamma&&& \end{array}\qquad \begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&\beta&&\\ 0&\beta&&&&\\ \gamma&&&&& \end{array}$$

$$\begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&\beta&&\\ 0&\gamma&&&&\\ \beta&&&&& \end{array}\qquad \begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&\beta&&\\ 0&&&&&\\ \beta&&&&&\\ \gamma&&&&& \end{array}\qquad \begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&&&\\ 0&\beta&\beta&&&\\ \gamma&&&&& \end{array}\qquad \begin{array}{llllll} 0&0&0&0&\alpha&\alpha\\ 0&0&0&&&\\ 0&\beta&&&&\\ \beta&&&&&\\ \gamma&&&&& \end{array}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha & \alpha \\ 0 & 0 & 0 \\ 0 & \beta \\ \beta & \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha & \beta \\ 0 & \beta & \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha & \beta \\ 0 & \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha & \beta \\ 0 & \gamma \\ \beta \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha & \beta \\ 0 \\ \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha \\ 0 & \beta & \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha \\ 0 & \beta \\ \beta & \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \alpha \\ 0 & \beta \\ \beta \\ \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \\ 0 & \alpha \\ \beta & \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \\ 0 & \alpha & \gamma \\ \beta \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \\ 0 & \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & \alpha & \beta \\ \beta & \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & \alpha & \beta \\ \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & \alpha \\ \beta & \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \\ 0 & \gamma \\ \alpha \\ \beta \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 & \beta \\ 0 \\ \alpha \\ \beta \\ \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & \beta \\ \alpha & \gamma \\ \beta \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 & \alpha \\ 0 & 0 & 0 \\ 0 & \beta \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & \alpha & \beta \\ \beta & \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & \alpha & \beta \\ \beta \\ \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & \alpha \\ \beta & \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & \beta \\ \alpha & \gamma \\ \beta \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \alpha \\ 0 & \beta \\ \alpha \\ \beta \\ \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha & \alpha \\ \beta & \beta \\ \gamma \end{bmatrix}$$

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha \\ \alpha & \beta \\ \beta & \gamma \end{bmatrix} \quad \begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & \alpha \\ \alpha & \beta \\ \beta \\ \gamma \end{bmatrix}$$

Hence we obtain

$$\begin{aligned}\{4\ 3\ 1\}\{2^2\ 1\} = {} & \{6\ 5\ 2\}+\{6\ 5\ 1^2\}+\{6\ 4\ 3\}+2\{6\ 4\ 2\ 1\}+\{6\ 4\ 1^3\}+ \\ & +\{6\ 3\ 2^2\}+\{6\ 3\ 2\ 1^2\}+\{6\ 3^2\ 1\}+\{5^2\ 3\}+2\{5^2\ 2\ 1\}+ \\ & +\{5^2\ 1^3\}+2\{5\ 4\ 3\ 1\}+2\{5\ 4\ 2^2\}+3\{5\ 4\ 2\ 1^2\}+ \\ & +\{5\ 4\ 1^4\}+\{5\ 3^2\ 2\}+\{5\ 3^2\ 1^2\}+2\{5\ 3\ 2^2\ 1\}+ \\ & +\{5\ 3\ 2\ 1^3\}+\{4^2\ 3\ 2\}+\{4^2\ 3\ 1^2\}+2\{4^2\ 2^2\ 1\}+ \\ & +\{4\ 3^2\ 2\ 1\}+\{4\ 3\ 2^3\}+\{4\ 3\ 2^2\ 1^2\}+\{4^2\ 2\ 1^3\}.\end{aligned}$$

## 6.4. Generating functions and further properties of $S$-functions (7)

### Convention for $S$-functions with parts in non-descending order (8)

In the expression of a partition $(\lambda)$ of $n$, the order of the parts is of no consequence. In the definition of the $S$-function

$$\{\lambda\} \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_p\},$$

however, it was assumed that the parts were expressed in descending order, i.e. $\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_p$.

It is convenient to define the $S$-function $\{\lambda\} \equiv \{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ when the above inequality does not hold, not by the simple rearrangement of the parts in descending order, but by the equation

$$\{\lambda_1, \lambda_2, \ldots, \lambda_p\} = |h_{\lambda_s - s + t}|.$$

The parts $\lambda_i$ must be integral, but need not be positive.

Every $S$-function so defined is either zero or equal to an $S$-function expressed with the parts in descending order, with a possible change of sign. To reduce such an $S$-function to this form we have the following three rules which are easily verified.

1. *In any $S$-function two consecutive parts may be interchanged provided that the preceding part is decreased by unity and the succeeding part increased by unity, the $S$-function being thereby changed in sign,* i.e.

$$\begin{aligned}&\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_i, \lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_p\} \\ &\qquad = -\{\lambda_1, \ldots, \lambda_{i-1}, \lambda_{i+1}-1, \lambda_i+1, \lambda_{i+2}, \ldots, \lambda_p\}.\end{aligned}$$

2. *In any $S$-function if any part exceed by unity the preceding part the value of the $S$-function is zero,* i.e. if

$$\lambda_{i+1} = \lambda_i+1,$$
$$\{\lambda\} = 0.$$

3. *The value of any S-function is zero if the last part is a negative number.*

EXAMPLES.

$$
\begin{aligned}
\{2,4,6,-1,2\} &= -\{2,5,5,-1,2\}\\
&= \{2,5,5,1,0\}\\
&= -\{4,3,5,1,0\}\\
&= \{4,4,4,1,0\}\\
&= \{4^3,1\}.\\
\{4,3,6,1,3\} &= -\{4,5,4,1,3\}\\
&= 0
\end{aligned}
$$

by reason of rule (2).

$$
\begin{aligned}
\{6,3,8,-2,1\} &= -\{6,3,8,0,-1\}\\
&= 0
\end{aligned}
$$

by reason of rule (3).

**Generalization of the definition of $S$-functions**

Hitherto we have associated the $S$-functions of $n$ quantities $\alpha_1, \alpha_2,\ldots, \alpha_n$ with two series, namely

$$f(x) = \prod (1-\alpha_r x) = 1-\{1\}x+\{1^2\}x^2-\ldots+(-1)^r\{1^r\}x^r+\ldots$$

and

$$F(x) = 1/\prod (1-\alpha_r x) = 1+\{1\}x+\{2\}x^2+\ldots+\{r\}x^r+\ldots.$$

Henceforward we choose the second series as the basic series. The $S$-functions are then regarded as those associated with the series $F(x)$. The concept of the $S$-function is then generalized as follows.

We take as the basic series any series of the form

$$F(x) = 1+\sum h_r x^r, \tag{6.4; 1}$$

irrespective of convergence or divergence. We define $S_r$ as the coefficient of $x^{r-1}$ in the formal quotient

$$F'(x)/F(x) = \sum S_r x^{r-1}, \tag{6.4; 2}$$

the dash denoting differentiation.

With these values for the $S_r$'s, the $S$-functions are defined by the usual formula

$$n!\{\lambda\} = \sum h_\rho \chi_\rho^{(\lambda)} S_\rho. \tag{6.4; 3}$$

The $S$-functions of weight $w$ do not depend on the values of $S_r$ for $r > w$, and hence depend only on the first $w$ coefficients $h_r$ in $F(x)$. Since a polynomial can always be found of which the reciprocal coincides with $F(x)$ for the first $w$ terms, all the properties of the $S$-functions so far demonstrated for the reciprocal of a polynomial must also hold for the general series $F(x)$, with the exception of those properties which involve the roots. In particular $\{r\} = h_r$.

Some special cases are of interest. In the first case we may retain the equation $F(x) = 1/\prod(1-\alpha_r x)$, but make the number of the roots $\alpha_r^{-1}$ become infinite, $\sum_r \alpha_r^i$ being supposed convergent for all $i$. In this case the relations between the $S$-functions and the quantities $\alpha_r$ are also applicable, the equations connecting the $S$-functions with symmetric functions of the form $\sum \alpha_1^{a_1}\alpha_2^{a_2}\ldots$ holding true. The equation expressing the $S$-function as a quotient of determinants then involves the limit of the quotient as the number of rows and columns becomes infinite.

Secondly, $F(x)$ may be of the form

$$F(x) = \prod(1-\alpha_r x).$$

In this case the functions $a_n$ and $h_n$ are interchanged, and each $S$-function with the conjugate $S$-function, there being a change of sign also when the weight is odd. The number of the quantities $\alpha_r$ may become infinite provided that $\sum \alpha_r^i$ is convergent for all $i$.

Thirdly, the rational fraction is important,

$$F(x) = \prod_{r=1}^{q}(1-\beta_r x)\Big/\prod_{r=1}^{p}(1-\alpha_r x).$$

In this case $$S_n = \sum \alpha_r^n - \sum \beta_r^n.$$

$p$ and $q$ may become infinite provided that $\sum \alpha_r^i$ and $\sum \beta_r^i$ are convergent for all $i$.

Lastly, the effect of an exponential factor is worth noticing. If

$$F(x) = e^{x^r},$$

then $$S_r = r, \qquad S_i = 0 \quad (i \neq r).$$

Hence if $$g(x) = a_1 x + a_2 x^2 + a_3 x^3 + \ldots$$

and $$F(x) = e^{g(x)} \prod(1-\beta_r x)/\prod(1-\alpha_r x),$$

then $$S_n = na_n + \sum \alpha_r^n - \sum \beta_r^n.$$

For a meromorphic function of genus $i$, for $n > i$, $S_n$ is the sum of the $n$th powers of the reciprocals of the poles minus the sum of the $n$th powers of the reciprocals of the zeros. The exponential factors may be adjusted to give any assigned values to $S_1, S_2, \ldots, S_i$.

### Generating functions in one variable

Given that $$F(x) = 1 + \sum h_r x^r,$$

the equation $$1/F(x) = 1 + \sum (-1)^r a_r x^r$$

may be regarded as a generating function for the quantities $a_r$. This immediately suggests the problem of finding generating functions for the other $S$-functions.

One difficulty is that the number of $S$-functions of weight $w$ is equal to the number of partitions of $w$, and a simple method of associating all partitions of all numbers with powers of a variable, or products of powers of variables, is not obvious, unless an infinite number of variables is taken. It is possible, however, to obtain generating functions if we restrict ourselves to partitions of a definite form.

Consider the $S$-functions

$$\{n,p_1,p_2,\dots,p_i\},$$

where $p_1, p_2,\dots, p_i$ are fixed numbers ($p_1 \geqslant p_2 \geqslant \dots \geqslant p_i$).

We have, for $n \geqslant p_1$,

$$\{n,p_1,p_2,\dots,p_i\} = \begin{array}{cccc} & h_{n+1}, & & h_{n+i} \\ h_{p_1-1}, & h_{p_1}, & \dots, & h_{p_1+i-1} \\ & & & \\ h_{p_i-i}, & h_{p_i-i+1}, & \dots, & h_{p_i} \end{array} \qquad (6.4;4)$$

Now consider the product $F(x).g(x)$, where

$$g(x) = \begin{array}{cccc} x^i, & x^{i-1}, & \dots, & 1 \\ h_{p_1-1}, & h_{p_1}, & \dots, & h_{p_1+i-1} \\ h_{p_2-2}, & h_{p_2-1}, & \dots, & h_{p_2+i-2} \\ & & & \\ h_{p_i-i}, & h_{p_i-i+1}, & \dots, & h_{p_i} \end{array} \qquad (6.4;5)$$

The coefficient of $x^{n+i}$, for $n \geqslant p_i$, is clearly $\{n,p_1,p_2,\dots,p_i\}$ from (6.4; 4). Hence $F(x).g(x)$ may be regarded as a generating function for $S$-functions of the form $\{n;p_1,\dots,p_i\}$.

I. $F(x).g(x) = P(x)+\sum\limits_{n=p_1}^{\infty}\{n,p_1,p_2,\dots,p_i\}x^{n+i}$, *where* $P(x)$ *is a polynomial of degree* $< p_1+i$.

The exact form of $P(x)$ is of interest. If $n < p_1+i$, the coefficient of $x^n$ is still equal to $\{n-i,p_1,\dots,p_i\}$ but the parts are not in descending order and the usual rule must be used to reduce them to descending order.

If we can find $j$ so that

$$n-i+j = p_j,$$

the coefficient of $x^n$ is clearly zero.

A special case is of particular interest. If

$$p_1 = p_2 = \dots = p_i = r+1,$$

then

$$P(x) = (-1)^i \sum_{n=0}^{r} \{r^i, n\} x^n,$$

$$F(x)g(x) = P(x)+O(x^{r+i+1}),$$

$$F(x) = P(x)/g(x)+O(x^{r+i+1}),$$

where $P(x)$ is a polynomial of degree $r$, and $g(x)$ a polynomial of degree $i$. The nomenclature $O(x^n)$ is used to denote any sum or infinite series of powers of $x$ for which the indices $\geqslant n$.

This is the rational fraction with numerator and denominator of degrees $r$ and $i$ respectively which, expressed as a power series, coincides with $F(x)$ for the greatest number of terms. It has been extensively used by Padé and others in the study of continued fractions (**9**).

A generalization of the Padé continued fraction presents itself.

II. *$P(x)$ and $g(x)$ may be chosen so that $g(x)$ is a polynomial of degree $i$, and $P(x)$ a polynomial of degree less than or equal to $i+r$, in which the coefficients of $i$ arbitrarily assigned powers of $x$ are zero, and*

$$F(x) = P(x)/g(x)+O(x^{i+r+1}).$$

If the missing indices are $\alpha_1, \alpha_2, \dots, \alpha_i$ $(\alpha_1 > \alpha_2 > \dots > \alpha_i)$, then take

$$p_j = \alpha_j - j + i,$$

and define $g(x)$ and $P(x)$ by (6.4; 5) and Theorem I.

The generating function for the conjugate $S$-function

$$\{q_1, q_2, \dots, q_i, 1^n\}$$

may be obtained from Theorem I, by replacing $f(x)$ by $1/f(x)$ and also replacing each term $h_r$ by $a_r$. The coefficients of odd powers of $x$ then carry an extra minus sign.

## Generating functions in more than one variable

Another generating function may be found which gives all the $S$-functions which correspond to partitions into not more than $p$ parts.

Consider the function of $p$ variables

$$\phi(x_1, x_2, \dots, x_p) = \prod_{r=1}^{p} F(x_r)\, \Delta(x_1, x_2, \dots, x_p),$$

where

$$\Delta(x_1, x_2, \dots, x_p) = \prod (x_r - x_s) \quad (r < s)$$
$$= |x_s^{p-t}|.$$

$\phi$ is an alternating function of the variables. Consider the coefficient of $x_1^{\lambda_1+p-1}x_2^{\lambda_2+p-2}\dots x_p^{\lambda_p}$ $(\lambda_1 \geqslant \lambda_2 \geqslant \dots \geqslant \lambda_p)$. There is a term in $\phi$ corresponding to each term in the expansion of the determinant for $\Delta(x_1,\dots,x_p)$. Let $x_1^{e_1}x_2^{e_2}\dots x_p^{e_p}$ be any term of this expansion. To obtain $\prod x_i^{\lambda_i+p-i}$ this term must be multiplied by $\prod x_i^{\lambda_i+p-i-e_i}$, and the coefficient is

$$\pm \prod h_{\lambda_i+p-i-e_i}.$$

Hence the coefficient of $\prod x_i^{\lambda_i+p-i}$ in $\phi$ is

$$|h_{\lambda_s-s+t}| \quad \text{or} \quad \{\lambda_1,\lambda_2,\dots,\lambda_p\}.$$

Hence we have

III.

$$\prod F(x_r)\,\Delta(x_1,x_2,\dots,x_p) = \sum \pm\{\lambda_1,\lambda_2,\dots,\lambda_p\}x_1^{\lambda_1+p-1}x_2^{\lambda_2+p-2}\dots x_p^{\lambda_p}.$$

The summation is taken for all partitions into not more than $p$ parts, and for all permutations of the suffixes of the $x$'s, the minus sign being taken for an odd permutation.

Replacing $F(x)$ by $1/F(x)$, we obtain the conjugate result, namely

IV.

$$\Delta(x_1,x_2,\dots,x_p)/\prod F(x_r)$$
$$= \sum \pm\{p^{\lambda_p},(p-1)^{\lambda_{p-1}-\lambda_p},\dots,1^{\lambda_1-\lambda_2}\}x_1^{\lambda_1+p-1}x_2^{\lambda_1+p-2}\dots x_p^{\lambda_p},$$

*where the summation on the right includes all partitions in which none of the parts exceed $p$ in magnitude. The term takes a minus sign for a negative permutation of the suffixes when the weight of the coefficient is even, and for a positive permutation when the weight is odd.*

An important variant of these two theorems is obtained as follows.

Denote $S$-functions of the quantities $x_1$, $x_2,\dots$, $x_n$ by

$$\{x;\,\lambda_1,\lambda_2,\dots,\lambda_p\} \quad \text{or} \quad \{x;\,\lambda\}.$$

Then we have

$$\sum \pm x_1^{\lambda_1+p-1}x_2^{\lambda_2+p-2}\dots x_p^{\lambda_p} = \{x;\,\lambda_1,\lambda_2,\dots,\lambda_p\}\Delta(x_1,\dots,x_p).$$

Thus, from Theorems III and IV,

$$\prod F(x_r)\Delta(x_1,\dots,x_p) = [1+\sum\{\lambda\}\{x;\,\lambda\}]\Delta(x_1,\dots,x_p),$$
$$\Delta(x_1,\dots,x_p)/\prod F(x_r) = [1\pm\sum\{\lambda\}\{x;\,\tilde{\lambda}\}]\Delta(x_1,\dots,x_p).$$

Hence we have

V.

$$\prod F(x_r) = 1+\sum\{\lambda\}\{x;\,\lambda\},$$
$$1/\prod F(x_r) = 1\pm\sum\{\lambda\}\{x;\,\tilde{\lambda}\}.$$

$(\tilde{\lambda})$ denotes here, and elsewhere in this book, the partition conjugate to $(\lambda)$. Of the alternative signs, the minus is taken when the weight is odd.

This result may be compared with one mentioned by MacMahon,[†] easily obtained by direct multiplication:

$$\prod F(x_r) = 1+\sum h_{(\lambda)} P_{(\lambda)},$$

where

$$h_{(\lambda)} = h_{\lambda_1} h_{\lambda_2} ... h_{\lambda_p},$$

and

$$P_{(\lambda)} = \sum x_1^{\lambda_1} x_2^{\lambda_2} ... x_p^{\lambda_p}.$$

If

$$F(x) = \prod (1-\alpha_r x)^{-1},$$

then

$$\prod F(x_r) = \prod_{r,s} (1-\alpha_r x_s)^{-1},$$

a product symmetrical with respect to the two sets of quantities $\alpha_1, \alpha_2,...,$ and $x_1, x_2,..., x_p$. This essential symmetry is brought out by the form of Theorem V, but not by the form given by MacMahon.

Again, comparing the two results, it follows that, if

$$\{\lambda\} = \sum K_{\lambda\mu} h_{(\mu)},$$

then

$$\sum \alpha_1^{\mu_1} \alpha_2^{\mu_2} ... = \sum K_{\lambda\mu} \{\lambda\}, \qquad (6.4; 6)$$

where the coefficients $K_{\lambda\mu}$ are the same in both equations.

This equality was noticed by Kostka,[‡] who constructed tables of the coefficients.

### $S$-functions associated with a product

Let

$$F(x) = f(x) g(x),$$

where

$$f(x) = 1+\sum h'_n x^n,$$

$$g(x) = 1+\sum h''_n x^n,$$

$$F(x) = 1+\sum h_n x^n.$$

Some formulae expressing the symmetric functions associated with $F(x)$, in terms of those associated with $f(x)$ and $g(x)$ respectively, are well known. Thus

$$h_n = \sum h'_r h''_{n-r},$$

$$a_n = \sum a'_r a''_{n-r},$$

and

$$S_n = S'_n + S''_n.$$

We now obtain a general formula expressing any $S$-function associated with $F(x)$ in terms of the $S$-functions of $f(x)$ and $g(x)$.

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[†] MacMahon, *Combinatory Analysis* (1915), vol. i, 55 et seq.

[‡] See Muir, *Theory of Determinants*, 4 (1923), 145.

Denote the $S$-functions of $F(x)$, $f(x)$, and $g(x)$ respectively by $\{\lambda\}$, $\{\lambda\}'$, and $\{\lambda\}''$.

Clearly $$\prod F(x_r) = \prod f(x_r) \prod g(x_r).$$

Hence, from Theorem V,

$$\begin{aligned} 1+\sum\{\lambda\}\{x;\lambda\} &= [1+\sum\{\lambda\}'\{x;\lambda\}][1+\sum\{\lambda\}''\{x;\lambda\}] \\ &= 1+\sum\{\lambda\}'\{\mu\}''\{x;\lambda\}\{x;\mu\} \\ &= 1+\sum g_{\lambda\mu\nu}\{\lambda\}'\{\mu\}''\{x;\nu\}, \end{aligned}$$

where $g_{\lambda\mu\nu}$ is the coefficient of $\{\nu\}$ in $\{\lambda\}\{\mu\}$.

Hence we have

VI. $$\{\nu\} = \sum g_{\lambda\mu\nu}\{\lambda\}'\{\mu\}''.$$

The coefficients $g_{\lambda\mu\nu}$ may be found by the multiplication theorem for $S$-functions.

Examples. The $S$-functions associated with $(1-x)^{-m}$ are known.

We have $$\{\lambda\} = \chi_0^{(\lambda)}(P_\lambda)/n!,$$

where $(P_\lambda)$ is the product of the first $\lambda_i$ terms from each $i$th row of the set of numbers

$$\begin{array}{lllll} m, & m+1, & m+2, & m+3, & \ldots \\ m-1, & m, & m+1, & m+2, & \ldots \\ m-2, & m-1, & m, & m+1, & \ldots \end{array}$$

and $(\lambda)$ is a partition of $n$.

By substituting in Theorem VI for the $S$-functions of

$$(1-x)^{-m-n} = (1-x)^{-m}(1-x)^{-n},$$

various identities may be obtained.

Denote the product $r(r+1)(r+2)\ldots(r+i-1)$ by $[r]_i$. When we put $\{\nu\} = \{p\}$, Theorem VI gives

$$[m+n]_p/p! = \sum_{r=0}^{p}[m]_r[n]_{p-r}/r!\,(p-r)!,$$

which is Vandermonde's theorem.

Putting $\{\nu\} = \{p^2\}$, we notice that $\{p^2\}$ appears only in products of the form $\{r,s\}\{p-s,p-r\}$, and then with coefficients unity.

Hence $$\{p^2\} = \sum\{r,s\}'\{p-s,p-r\}''.$$

We have $$\chi_0^{(r,s)} = (r+s)!(r-s+1)/(r+1)!\,s!,$$

whence, from Theorem VI,

$$\frac{[m+n]_p[m+n-1]_p}{(p+1)!\,p!} = \sum \frac{[m]_r[m-1]_s[n]_{p-s}[n-1]_{p-r}(r-s+1)^2}{(r+1)!\,s!\,(p-s+1)!\,(p-r)!}.$$

This may be simplified, and, on replacing $m$, $n$, $p$ by $x$, $y$, $n$, we may write it in the form

$$\begin{aligned}
&[x+y]_n[x+y-1]_n \\
&\quad = [y]_n[y-1]_n+\frac{n(n+1)}{2.1}[x]_1[x-1]_1[y]_{n-1}[y-1]_{n-1}+ \\
&\qquad\qquad +\frac{n(n-1)(n+1)n}{2.3.1.2}[x]_2[x-1]_2[y]_{n-2}[y-1]_{n-2}+\ldots+ \\
&\qquad +2^2\Big[\tfrac{1}{2}n[x]_1[y]_n[y-1]_{n-1} \\
&\qquad\qquad +\frac{n(n-1)(n+1)}{2.3.1}[x]_2[x-1]_1[y]_{n-1}[y-1]_{n-2}+\ldots\Big]+ \\
&\qquad +3^2\Big[\frac{n(n-1)}{2.3}[x]_2[y]_n[y-1]_{n-2} \\
&\qquad\qquad +\frac{n(n-1)(n-2)(n+1)}{2.3.4.1}[x]_3[x-1]_1[y]_{n-1}[y-1]_{n-3}+\ldots\Big]+ \\
&\qquad + \quad . \quad . \quad . \quad . \quad + \\
&\qquad +(n+1)[x]_n[y]_n.
\end{aligned}$$

Picking out the coefficient of $x^r y^{2n-r}$ in this and denoting

$$x(x-1)(x-2)\ldots(x-i+1)$$

by $(x)_i$, we obtain

$$\begin{aligned}
\frac{r(2n)_{r-1}}{(n)_{r-1}} &= r^2+\frac{r(n+1)}{(n-r+2).1}(r-2)^2+ \\
&\qquad +\frac{r(r-1)(n+1)n}{(n-r+2)(n-r+3).1.2}(r-4)^2+ \\
&\qquad +\frac{r(r-1)(r-2)(n+1)n(n-1)}{(n-r+2)(n-r+3)(n-r+4)1.2.3}(r-6)^2+ \\
&\qquad +\ldots \quad \text{to } \tfrac{1}{2}r \text{ or } \tfrac{1}{2}(r+1) \text{ terms.}
\end{aligned}$$

A generalization of the above to products of the form

$$[x+y]_n[x+y-1]_n\ldots[x+y-i+1]_n$$

is easily obtained by considering the $S$-function $\{p^i\}$. We obtain on the right an $i$-dimensional series.

Another interesting identity is obtained by taking

$$\{\nu\} = \{1+p, 1^q\}.$$

$\{\nu\}$ appears with coefficient unity in products of the form

$$\{1+r, 1^s\}\{p-r, 1^{q-s}\} \quad \text{and} \quad \{1+r, 1^s\}\{1+p-r, 1^{q-s-1}\}.$$

Hence, from Theorem VI,

$$\{1+p, 1^q\} = \{1+p, 1^q\}'+\{1+p, 1^q\}''+$$
$$+\sum \{1+r, 1^s\}'\{p-r, 1^{q-s}\}''+\sum \{1+r, 1^s\}'\{1+p-r, 1^{q-s-1}\}''.$$

Remembering that $\chi_0^{(1+p,1^q)} = (p+q)!/p!\,q!$, we deduce the identity

$$\frac{[x+y-q]_{p+q+1}}{(p+q+1)p!\,q!}$$
$$= \frac{[x-q]_{p+q+1}}{(p+q+1)p!\,q!}+y\sum_{rs}{}' \frac{[x-s]_{r+s+1}[y-q+s+1]_{p+q-r-s-1}}{(r+s+1)r!\,s!\,(p-r)!\,(q-s)!}+$$
$$+\frac{[y-q]_{p+q+1}}{(p+q+1)p!\,q!}.$$

$\sum'$ excludes the case $r = p$, $s = q$.

One deduction from Theorem VI is of particular interest. If $g(x) = 1/f(x)$, then $F(x) = f(x)g(x) = 1$. Clearly all the $S$-functions of $F(x)$ are zero. If we denote an $S$-function of $f(x)$ by $\{\lambda\}$, Theorem VI gives, for any fixed $S$-function $\{\nu\}$,

$$\sum (-1)^w g_{\lambda\mu\nu}\{\lambda\}\{\tilde{\mu}\} = 0,$$

where $w$ is the weight of $\{\mu\}$.

If $\{\nu\} = \{p\}$, this gives the well-known result

$$h_p-h_{p-1}a_1+h_{p-2}a_2-h_{p-3}a_3+\ldots\pm a_p = 0.$$

Another result analogous to this may be obtained as follows.

Let $$\phi = \prod f(\rho x_r) = 1+\sum \{\lambda\}\{x;\, \lambda\}\rho^n,$$

where $(\lambda)$ is a partition of $n$.

Then $$\partial\phi/\partial\rho = \sum n\{\lambda\}\{x;\, \lambda\}\rho^{n-1}.$$

But $$1/\phi = \sum (-1)^m\{\tilde{\mu}\}\{x;\, \mu\}\rho^m,$$

where $(\mu)$ is a partition of $m$. Consequently

$$\frac{1}{\phi}\frac{\partial\phi}{\partial\rho} = \sum (-1)^m n\{\lambda\}\{\tilde{\mu}\}\{x;\, \lambda\}\{x;\, \mu\}\rho^{m+n-1}$$
$$= \sum (-1)^m n g_{\lambda\mu\nu}\{\lambda\}\{\tilde{\mu}\}\{x;\, \nu\}\rho^{m+n-1}.$$

Again, $\log\phi = \sum \log f(\rho x_r)$,

and, by differentiating,

$$\begin{aligned}\frac{1}{\phi}\frac{\partial\phi}{\partial\rho} &= \sum \frac{x_r}{f(\rho x_r)}\frac{\partial f(\rho x_r)}{\partial(\rho x_r)} \\ &= \sum_{rn} S_n x_r^n \rho^{n-1} \\ &= \sum_{n} S_n Z_n \rho^{n-1} \\ &= \sum (-1)^q S_n\{x; p, 1^q\}\rho^{n-1},\end{aligned}$$

summed for all partitions of the form $(p, 1^q)$, where $p+q = n$, $Z_n = \sum x_r^n$.

Hence

$$\begin{aligned}\sum (-1)^m n g_{\lambda\mu\nu}\{\lambda\}\{\tilde{\mu}\} &= -(1)^q S_k \quad \text{if } (\nu) = (p, 1^q), \\ &= 0 \quad \text{otherwise},\end{aligned}$$

where $(\lambda)$ is a partition of $n$, $(\mu)$ a partition of $m$, and $(\nu)$ a partition of

$$k = m+n = p+q.$$

### Isobaric determinants

In the expression of the $S$-function as a determinant,

$$\{\lambda\} = |h_{\lambda_s - s+t}|,$$

the suffixes of the $h$'s increase by unity in passing from one column to the next, but they decrease by arbitrary steps, depending on the particular $S$-function chosen, in passing from one row to the next.

The more general determinant in which the suffixes in the columns differ by arbitrary leaps, as well as in the rows, has received considerable attention from Naegelsbach† and others, and Aitken‡ has given a theorem of duality connecting such determinants with the corresponding determinants involving the $a_r$'s.

Provided that the leap in the suffixes on passing from one column to the next is the same in each row, all terms in the development of the determinant will have equal weight. Hence they are termed *isobaric determinants*.

We now consider the theory of these determinants, obtaining a generating function for them, the expression for them as a sum of $S$-functions, and a proof of Aitken's theorem.

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† See Muir, *Theory of Determinants*, **3** (1920), 144.

‡ Aitken, *Proc. Edinburgh Math. Soc.* (2), **1** (1929), 55, and (2), **2** (1931), 164.

The determinant may be written

$$\{\lambda/\mu\} = |h_{\lambda_s-\mu_t-s+t}|,$$

where $(\lambda)$ denotes the partition $(\lambda_1,\lambda_2,\ldots,\lambda_n)$ and $(\mu)$ the partition $(\mu_1,\mu_2,\ldots,\mu_n)$, $(\lambda_1 \geqslant \lambda_2 \geqslant \ldots \geqslant \lambda_n,\ \mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_n)$, the determinant having $n$ rows and columns.

Consider the determinant with $n$ rows and columns,

$$\phi = \left|\frac{x_s^p f(x_s)-y_t^p f(y_t)}{x_s-y_t}\right.$$

This is an alternating function of $x_1,x_2,\ldots,x_n$ and also of $y_1, y_2,\ldots, y_n$. The coefficient of $\prod x_r^{\lambda_r+n-r}\, y_{n-r}^{\alpha_r+n-r}$ is clearly

$$|h_{\lambda_s+\alpha_{n-t}+n-s+t-p+1}|,$$

or, if we put $\mu_t = q-\alpha_{n-t}$,

$$|h_{\lambda_s-\mu_t-s+t+n+q-p+1}| = |h_{\lambda_s-\mu_t-s+t}|,$$

for the special value $q = p-n-1$.

This result is unaffected if all the numbers $\lambda_r$ and $\mu_r$ are increased or decreased by the same amount. Hence

VII. *The coefficient of* $\prod x_r^{\lambda_r+n-r-k} y_r^{p-1-\mu_r-n+r+k}$ *in*

$$\phi = \left|\frac{x_s^p f(x_s)-y_t^p f(y_t)}{x_s-y_t}\right|$$

*is* $|h_{\lambda_s-\mu_t-s+t}| = \{\lambda/\mu\}$.

Thus $\phi$ may be regarded as a generating function for the quantities $\{\lambda/\mu\}$.

Now the coefficient of $\prod y_r^{p-1-\mu_r-n+r}$ is clearly

$$|x_s^{\mu_t+n-t} f(x_s)|,$$

which is a generating function for the quantities $\{\lambda/\mu\}$ when $(\mu)$ is fixed.

$$\begin{aligned}
|x_s^{\mu_t+n-t} f(x_s)| &= \prod f(x_r)\,|x_s^{\mu_t+n-t}| \\
&= [1+\sum \{\nu\}\{x;\nu\}]\{x;\mu\}\Delta(x_1,\ldots x_n) \\
&= \sum g_{\nu\mu\lambda}\{\nu\}\{x;\lambda\}\Delta(x_1,\ldots,x_n) \\
&= \sum g_{\nu\mu\lambda}\{\nu\} \prod x_r^{\lambda_r+n-r},
\end{aligned}$$

where $g_{\nu\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\{\mu\}$.

Hence the coefficient of $\prod x_r^{\lambda_r+n-r} y_r^{p-1-\mu_r-n+r}$ in $\phi$ is $\sum g_{\nu\mu\lambda}\{\nu\}$. Using Theorem VII, we obtain

VIII. $$\{\lambda/\mu\} = |h_{\lambda_s-\mu_t-s+t}| = \sum g_{\nu\mu\lambda}\{\nu\}.$$

As an example of this theorem consider the determinant

$$\begin{array}{ccc} h_4, & h_5, & h_7 \\ h_2, & h_3, & h_5 \\ 1, & h_1, & h_3 \end{array}$$

This may be expressed as $|h_{\lambda_s-\mu_t-s+t}|$ if $(\lambda) = (5, 4, 3)$ and $(\mu) = (1^2)$.

Now $\{5, 4, 3\}$ appears in the product $\{\nu\}\{1^2\}$ with coefficient unity if either
$$\{\nu\} = \{4, 3^2\}, \{4^2, 2\}, \text{ or } \{5, 3, 2\}.$$

Hence $$\begin{vmatrix} h_4, & h_5, & h_7 \\ h_2, & h_3, & h_5 \\ 1, & h_1, & h_3 \end{vmatrix} = \{4, 3^2\}+\{4^2, 2\}+\{5, 3, 2\}.$$

Aitken's theorem may now be deduced immediately. The conjugate $S$-functions, being the $S$-functions of the series $1/f(-x)$, obviously obey the same multiplication law as the $S$-functions themselves. Hence $g_{\nu\mu\lambda}$ is also the coefficient of $\{\tilde{\lambda}\}$ in the product $\{\tilde{\nu}\}\{\tilde{\mu}\}$. Thus, denoting conjugate partitions and $S$-functions by a tilde, we have

IX. $$\{\tilde{\lambda}/\tilde{\mu}\} = \sum g_{\nu\mu\lambda}\{\tilde{\nu}) = |a_{\lambda_s-\mu_t-s+t}|.$$

This, expressed differently, is Aitken's theorem.

### A third type of generating function for $S$-functions

We now obtain a generating function for all $S$-functions which correspond to partitions in which not more than $m$ parts are greater than $n$ in magnitude.

The simplest case, when $m = n = 1$, presents no difficulty.

X. $$\frac{f(x)-f(y)}{(x-y)f(y)} = \sum(-1)^\beta\{1+\alpha, 1^\beta\}x^\alpha y^\beta.$$

Clearly,
$$[f(x)-f(y)]/(x-y) = \sum h_r(x^{r-1}+x^{r-2}y+\dots+y^{r-1}).$$

Hence the coefficient of $x^\alpha y^\beta$ in
$$[f(x)-f(y)]/(x-y)f(y)$$
$$= [\sum h_r(x^{r-1}+x^{r-2}y+\dots+y^{r-1})][1+\sum(-1)^i a_i y^i]$$
is $$(-1)^\beta[h_{\alpha+1}a_\beta-h_{\alpha+2}a_{\beta-1}+\dots\pm h_{\alpha+\beta+1}] = (-1)^\beta\{1+\alpha, 1^\beta\};$$
this proves the theorem.

We next consider the general case when $m = n$. We require $2m$ variables

$$x_1, x_2,\ldots, x_m; \qquad y_1, y_2,\ldots, y_m,$$

and obtain a generating function for

$$\sum \pm\{1+\lambda_1,\ldots, m+\lambda_m, m^{\mu_m}, (m-1)^{\mu_{m-1}-\mu_m-1},\ldots, 1^{\mu_1-\mu_2-1}\} \prod x_r^{\lambda_r} y_r^{\mu_r}. \tag{6.4; 7}$$

Expressing the $S$-function, given as the coefficient of the typical term in (6.4; 7), as a determinant in terms of the $h_r$'s, we consider the Laplace development in terms of the first $m$ rows.

The minor obtained from the first $m$ rows and the $i_1$th, $i_2$th,..., $i_m$th columns is

$$|h_{\lambda_s+i_t}|.$$

The conjugate minor is a determinant of the form

$$|h_{\alpha_s-\beta_t-s+t}|,$$

where $\beta_t$ is the number of terms $i_j-j$ that are greater than or equal to $t$, and $\alpha_s$ is the number of terms $\mu_j-m+j$ greater than or equal to $s$.

From Theorem IX (Aitken's theorem),

$$|h_{\alpha_s-\beta_t-s+t}| = |a_{\mu_s-i_t+1}|.$$

Hence

$$\{1+\lambda_1, 2+\lambda_2,\ldots, m+\lambda_m, m^{\mu_m}, (m-1)^{\mu_{m-1}-\mu_m-1},\ldots, 1^{\mu_1-\mu_2-1}\}$$
$$= \sum \pm |h_{\lambda_s+i_t}|\,|a_{\mu_s-i_t+1}|.$$

Now consider the determinant

$$\frac{|f(x_s)-f(y_t)|}{x_s-y_t}$$

The coefficient of $\prod x_r^{\lambda_r} y_r^{i_r-1}$ is $|h_{\lambda_s+i_t}|$.

Hence the coefficient of $\prod x_r^{\lambda_r} y_r^{\mu_r}$ in

$$\left|\frac{f(x_s)-f(y_t)}{x_s-y_t}\right. \quad \prod f(y_r)$$

is $\quad \sum \pm |h_{\lambda_s+i_t}|\,|a_{\mu_s-i_t+1}| = \pm\{1+\lambda_1,\ldots, m+\lambda_m, m^{\mu_m},\ldots, 1^{\mu_1-\mu_2-1}\}$.

XI. $$\left.\frac{{}_s)-f(y_t)}{x_s-y_t}\right| \Big/ \prod f(y_r)$$
$$= \sum \pm\{1+\lambda_1,\ldots, m+\lambda_m, m^{\mu_m},\ldots, 1^{\mu_1-\mu_2-1}\} \prod x_r^{\lambda_r} y_r^{\mu_r}.$$

Throughout these equations when alternative signs are indicated, the factor $-1$ is introduced for a negative permutation of the suffixes of the $x$'s, or of the $y$'s, and again if the total degree in the $y$'s decreased by $\frac{1}{2}n(n-1)$ is odd.

The left-hand side of the equation of Theorem XI may be written

$$\left|\frac{f(x_s)-f(y_t)}{(x_s-y_t)f(y_t)}\right|,$$

which by Theorem X is equal to

$$|\sum(-1)^{\beta}\{1+\alpha, 1^{\beta}\}x_s^{\alpha}y_t^{\beta}|.$$

Picking out the coefficient of $\prod x_r^{\lambda_r}y_r^{\mu_r}$, and equating to the corresponding coefficient on the right-hand side of Theorem XI, we obtain

XII.

$$\{1+\lambda_1, 2+\lambda_2,\dots, m+\lambda_m, m^{\mu_m}, (m-1)^{\mu_{m-1}-\mu_m-1},\dots, 1^{\mu_1-\mu_2-1}\}$$
$$= |\{1+\lambda_s, 1^{\mu_t}\}|.$$

This expresses any $S$-function as a determinant in which the elements are $S$-functions of the form $\{p, 1^q\}$, and may be compared with the formulae expressing $S$-functions as determinants in which the elements are respectively of the form $h_r$ or $\{r\}$, and $a_r$ or $\{1^r\}$.

For example,

$$\{4, 3, 2^2, 1\} = \begin{array}{ccccc} h_4, & h_5, & h_6, & h_7, & h_8 \\ h_2, & h_3, & h_4, & h_5, & h_6 \\ 1, & h_1, & h_2, & h_3, & h_4 \\ & 1, & h_1, & h_2, & h_3 \\ & & & 1, & h_1 \end{array} \qquad \begin{array}{cccc} a_5, & a_6, & a_7, & \\ a_3, & a_4, & a_5, & \\ 1, & a_1, & a_2, & \\ & & 1, & a_1 \end{array}$$

$$= \begin{vmatrix} \{4, 1^4\}, & \{2, 1^4\} \\ \{4, 1^2\}, & \{2, 1^2\} \end{vmatrix} \qquad (6.4;8)$$

A further generalization is obtained later.

It is of interest to substitute in Theorem XII the known $S$-functions of the function $f(x) = (1-x)^{-m}$. The factors involving $m$, $m+1$, $m-1$, etc., may be taken out of the columns and rows, and correspond exactly on the two sides of the equation, but we are led to a formula for $\chi_0$, the degree of the character corresponding to the $S$-function.

$$\frac{\chi_0}{n!} \qquad \frac{}{\lambda_s!\,\mu_t!\,(\lambda_s+\mu_t+1)}$$

$$\chi_0 = \frac{n!}{\mu_1!\,\mu_2!\dots\mu_m!\,\lambda_1!\,\lambda_2!\dots\lambda_m!} \quad \frac{}{\lambda_s+\mu_t+1}$$

$$= \frac{n!\,\Delta(\lambda_1,\dots,\lambda_m)\Delta(\mu_1,\dots,\mu_m)}{\mu_1!\,\mu_2!\dots\mu_m!\,\lambda_1!\,\lambda_2!\dots\lambda_m!\prod(\lambda_r+\mu_p+1)}. \qquad (6.4;9)$$

This formula was obtained otherwise by Frobenius.†

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† Frobenius( 33), p. 516.

The case still remains to be considered of the generating function for $S$-functions which correspond to partitions in which not more than $m$ parts are greater than $n$ in magnitude, and $m \neq n$.

Consider first $m > n$, say $m = n+\alpha$. We take $n+m$ variables

$$x_1, x_2,\ldots, x_m; \qquad y_1, y_2, \quad y_n.$$

Denote by $\Delta_{mn}$ the determinant

$$\Delta_{mn} = \; x_s^{\alpha-t'} f(x_s) \,\vdots\, \frac{f(x_s)-f(y_{t''})}{x_s-y_{t''}}$$

in which $s$ runs from 1 up to $m$, $t'$ from 1 up to $\alpha$, and $t''$ from 1 up to $n$.

The coefficient of $\prod x_r^{\lambda_r} y_r^{i_r}$ is clearly

$$|h_{\lambda_s-\alpha+t'} \,\vdots\, h_{\lambda_s+i_{t''}}|.$$

Thus the coefficient of $\prod x_s^{\lambda_s} y_t^{\mu_t}$ in the quotient

$$\Delta_{mn}/\prod f(y_r)$$

is

$$\sum \pm |h_{\lambda_s-\alpha+t'} \,\vdots\, h_{\lambda_s+i_{t''}}| \, |a_{\mu_s-i_t+1}|$$
$$= \{\lambda_1-\alpha+1,\ \lambda_2-\alpha+2,\ldots,\ \lambda_m+n,\ n^{\mu_n}, (n-1)^{\mu_{n-1}-\mu_n-1},\ldots,\ 1^{\mu_1-\mu_2-1}\}.$$

Hence we have

XIII.

$$x_s^{\alpha-t'} f(x_s) \,\vdots\, \frac{f(x_s)-f(y_{t''})}{x_s-y_{t''}} \quad \prod f(y_r)$$
$$= \sum \pm \{\lambda_1-\alpha+1,\ \lambda_2-\alpha+2,\ldots,\ \lambda_m+n,\ n^{\mu_n},\ldots,\ 1^{\mu_1-\mu_2-1}\} \prod x_r^{\lambda_r} y_r^{\mu_r}.$$

The case in which $n > m$ can be obtained from this by substituting $1/f(x)$ for $f(x)$. Put $\alpha = n-m$.

XIV.

$$y_s^{\alpha-t'} \,\vdots\, \frac{f(x_{t''})-f(y_s)}{x_{t''}-y_s} \Big| \Big/ \prod f(y_r)$$
$$= \sum \pm \{\lambda_1+\alpha+1,\ \lambda_2+\alpha+2,\ldots,\ \lambda_m+n,\ n^{\mu_n},\ldots,\ 1^{\mu_1-\mu_2-1}\} \prod x_r^{\lambda_r} y_r^{\mu_r}.$$

Theorems XIII and XIV are a generalization of Theorems III, IV, X, and XI, which may be obtained from these by putting respectively

| | |
|---|---|
| $n = 0$ | in Theorem XIII, |
| $m = 0$ | in Theorem XIV, |
| $m = n = 1$ | in either theorem, |
| $m = n$ | in either theorem. |

In the same manner in which Theorem XII is deduced from Theorem XI we obtain

XV.

$$\{\lambda_1-\alpha+1,\ \lambda_2-\alpha+2,\ldots,\ \lambda_m+n,\ n^{\mu_n},\ (n-1)^{\mu_{n-1}-\mu_n-1},\ldots,\ 1^{\mu_1-\mu_2-1}\}$$
$$= |\{\lambda_s-\alpha+t'\}\,\vdots\,\{1+\lambda_s,\ 1^{\mu_{t'}}\}|.$$

XVI.

$$\{\lambda_1+\alpha+1,\ \lambda_2+\alpha+2,\ldots,\ \lambda_m+n,\ n^{\mu_n},\ (n-1)^{\mu_{n-1}-\mu_n-1},\ldots,\ 1^{\mu_1-\mu_2-1}\}$$
$$= |\{1^{\mu_s-\alpha+t'}\}\,\vdots\,\{1+\lambda_{t'},\ 1^{\mu_s}\}|.$$

As an example, besides the determinant forms given in (6.4; 8), $\{4,\ 3,\ 2^2,\ 1\}$ may be expressed

$$\{4,\ 3,\ 2^2,\ 1\} = \begin{array}{ccc} a_5, & a_6, & \{2,\ 1^6\} \\ {}_3, & a_4, & \{2,\ 1^4\} \\ 1, & a_1, & \{2,\ 1\} \end{array} \qquad \begin{array}{cccc} h_4, & h_5, & h_6, & \{7,\ 1\} \\ h_2, & h_3, & h_4, & \{5,\ 1\} \\ 1, & h_1, & h_2, & \{3,\ 1\} \\ & 1, & h_1, & \{2,\ 1\} \end{array}$$

By considering the $S$-functions associated with $(1-x)^{-m}$, we obtain a generalization of Frobenius's formula (6.4; 7).

XVII. *If a partition* $(\zeta)$ *of* $w$ *is expressed in any way in the form*

$$\{\lambda_1+\alpha+1,\ \lambda_2+\alpha+2,\ldots,\ \lambda_m+n,\ n^{\mu_n},\ (n-1)^{\mu_{n-1}-\mu_n-1},\ldots,\ 1^{\mu_1-\mu_2-1}\},$$

*where* $\alpha = n-m$ *is a positive, zero, or negative integer, then*

$$\chi_0^{(\zeta)} = \frac{w!\,\Delta(\lambda_1,\ldots,\lambda_m)\,\Delta(\mu_1,\ldots,\mu_n)}{\prod\lambda_r!\prod\mu_r!\prod(\lambda_r+\mu_s+1)}.$$

**Extension to rational fractions of the formula for the $S$-function as a quotient of determinants**

As in (6.3;1), if $f(x) = 1\Big/\prod\limits_{r=1}^{q}(1-\alpha_r x)$, then the corresponding $S$-functions $\{\lambda\}$ satisfy

$$\{\lambda\} = \frac{|\alpha_s^{\lambda_t+q-t}|}{|\alpha_s^{q-t}|}.$$

Again, if $g(x) = \prod\limits_{r=1}^{p}(1-\beta_r x)$, and $(\tilde{\mu})$ is the conjugate partition of $(\mu)$, then the corresponding $S$-functions satisfy

$$\{\mu\} = \frac{|\beta_s^{\tilde{\mu}_t+p-}}{|\beta_s^{p-t}|}.$$

We seek an extension of these results to the case when

$$F(x) = \prod_1^p(1-\beta_r x)\Big/\prod_1^q(1-\alpha_r x).$$

Denote $S$-functions of $F(x)$, $f(x)$, and $g(x)$ respectively by $\{\lambda\}$, $\{\lambda\}'$, and $\{\lambda\}''$. Then, from Theorem VI, we have

$$\{\nu\} = \sum g_{\lambda\mu\nu}\{\lambda\}'\{\mu\}'',$$

$$\{\nu\}\Delta(\alpha_1,\dots,\alpha_q)\Delta(\beta_1,\dots,\beta_p) = \sum g_{\lambda\mu\nu}|\alpha_s^{\lambda_t+q-t}||\beta_s^{\tilde{\mu}_t+p-t}|.$$

Consider first the particular case when $p = q = 2$, and $\{\nu\} = \{2^2\}$. We have

$$\{\nu\}(\alpha_1-\alpha_2)(\beta_1-\beta_2) = \begin{matrix} \alpha_1^3, & \\ \alpha_2^3, & \alpha_2^2 \end{matrix}\; \begin{matrix} \beta_1, & \\ \beta_2, & \end{matrix}\qquad \begin{matrix} \alpha_1^3, & \alpha_1 \\ & \end{matrix}\; \begin{matrix} \beta_1^2, \\ \beta_2^2, \end{matrix}\quad +$$

$$+\begin{vmatrix} \alpha_1^3, & 1 \\ \alpha_2^3, & 1 \end{vmatrix}\begin{vmatrix} \beta_1^2, & \beta_1 \\ \beta_2^2, & \beta_2 \end{vmatrix}+\begin{vmatrix} \alpha_1^2, & \alpha_1 \\ \alpha_2^2, & \alpha_2 \end{vmatrix}\begin{vmatrix} \beta_1^3, & 1 \\ \beta_2^3, & 1 \end{vmatrix}$$

$$\begin{vmatrix} \alpha_1^2 & 1 \\ \alpha_2^2, & 1 \end{vmatrix}\begin{vmatrix} \beta_1^3, & \beta_1 \\ \beta_2^3, & \beta_2 \end{vmatrix}+\begin{vmatrix} \alpha_1, & 1 \\ \alpha_2, & 1 \end{vmatrix}\begin{vmatrix} \beta_1^3, & \beta_1^2 \\ \beta_2^3, & \beta_2^2 \end{vmatrix}$$

$$\begin{matrix} \alpha_1^3, & \alpha_1^2, & \alpha_1, & 1 \\ \alpha_2^3, & \alpha_2^2, & \alpha_2, & 1 \\ \beta_1^3, & \beta_1^2, & \beta_1, & 1 \\ \beta_2^3, & \beta_2^2, & \beta_2, & 1 \end{matrix}$$

Thus $$\{\nu\} = \Delta(\alpha_1,\alpha_2,\beta_1,\beta_2)/\Delta(\alpha_1,\alpha_2)\Delta(\beta_1,\beta_2).$$

This result is easily extended to the case

$$F(x) = \prod_1^p(1-\beta_r x)\Big/\prod_1^q(1-\alpha_r x),$$

for the $S$-function $\{p^q\}$.

Clearly

$$\{p^q\} = \sum\{\lambda_1,\lambda_2,\dots,\lambda_q\}'\{p-\lambda_q,p-\lambda_{q-1},\dots,p-\lambda_1\}''.$$

Hence

$$\{p^q\}\Delta(\alpha_1,\dots,\alpha_q)\Delta(\beta_1,\dots,\beta_p) = \sum(-1)^{pq+\lambda_1+\dots+\lambda_q}|\alpha_s^{\lambda_t+q-t}||\beta_s^{\mu_t+q-t}|,$$

where $(\mu)$ is the partition conjugate to $(p-\lambda_q,p-\lambda_{q-1},\dots,p-\lambda_1)$. $(\mu)$ may be expressed in the form

$$\{q^{p-\lambda_1},(q-1)^{\lambda_1-\lambda_2},(q-2)^{\lambda_2-\lambda_3},\dots\},$$

whence it is clear that the indices in the second determinant are the indices omitted from the first determinant. Hence

XVIII. *For the function*

$$F(x) = \prod_1^p(1-\beta_r x)\Big/\prod_1^q(1-\alpha_r x),$$

$$\{p^q\}\Delta(\alpha_1,\dots,\alpha_q)\Delta(\beta_1,\dots,\beta_p) = \Delta(\alpha_1,\dots,\alpha_q,\beta_1,\dots,\beta_p).$$

Another proof of this theorem could be obtained by showing that, if the partition $(\lambda)$ has $q$ parts greater than or equal to $p$ in magnitude, and

$$F(x) = \prod_1^{p-1}(1-\beta_r x)\Big/\prod_1^{q-1}(1-\alpha_r x),$$

then $$\{\lambda\} = 0,$$

for $$\{\lambda\} = \sum g_{\nu\mu\lambda}\{\nu\}'\{\mu\}'',$$

and, since an $S$-function of $p-1$ quantities corresponding to a partition into $p$ or more parts is zero,† for each term in the sum either $\{\nu\}' = 0$ or $\{\mu\}'' = 0$. Hence, if $\alpha_r = \beta_s$, then $\{p^q\}$ must be zero, and $\{p^q\}$ contains a factor $\prod(\alpha_r-\beta_s)$. Elementary considerations complete the proof of the theorem.

More generally, it has been found possible to express any $S$-function corresponding to a partition into equal parts, of a rational fraction, as a quotient of determinants, as the following three theorems will show.

Consider first the case when $\{\nu\} = \{p^q\}$, and

$$F(x) = \prod_{r=1}^{p+i}(1-\beta_r x)\Big/\prod_{r=1}^{q+i}(1-\alpha_r x).$$

An exact repetition of the reasoning which leads to Theorem XVIII enables us to show that

$$\{p^q\}\Delta(\alpha_1,\ldots,\alpha_{q+1})\Delta(\beta_1,\ldots,\beta_{p+i}) = \sum(-1)^{pq+\lambda_1+\ldots+\lambda_q}|\alpha_s^{\lambda_t+q-t}||\beta_s^{\mu_t+q-t}|.$$

In the product of determinants on the right, the indices are taken from the numbers

$$p+q+i-1,\ p+q+i-2,\ldots,\ 3,\ 2,\ 1,\ 0.$$

The first $p+q$ numbers appear as indices in one of the determinants only, and the last $i$ numbers appear as indices in both determinants. Hence we have

XIX. *If* $$F(x) = \prod_{r=1}^{p+i}(1-\beta_r x)\Big/\prod_{r=1}^{q+i}(1-\alpha_r x),$$

*then the corresponding $S$-functions satisfy*

$$\{p^q\}\Delta(\alpha_1,\ldots,\alpha_{q+i})\Delta(\beta_1,\ldots,\beta_{p+i}) = \left|\begin{array}{c:c:c} \alpha_s^{p+q+i-t} & \alpha_s^{i-t'} & \\ \hdashline \beta_s^{p+q+i-t} & 0 & \beta_s^{i-t''} \end{array}\right|,$$

*in which $t$ runs from* 1 *to $p+q$, and $t'$ and $t''$ run from* 1 *to $i$.*

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† See p. 91.

There remains the case when the difference between the number of poles and the number of zeros of the rational fraction is not equal to the difference between the number of parts and the magnitude of each part in the partition. The results for this case are easily deduced from Theorem XIX.

First put $\beta_{p+i} = 0$. The last row contains only unity in the last column and zeros elsewhere. The determinant is equal to the minor of this element in the last row, last column. The factors $\beta_1, \beta_2, \ldots, \beta_{p+i-1}$, which will no longer appear in the product of the differences of the $\beta$'s if $\beta_{p+i}$ is omitted, may be taken out as a factor from the corresponding rows. This process may be repeated if more of the quantities $\beta_r$ are zero. Hence

XX. *If* $$F(x) = \prod_{r=1}^{p+i-j} (1-\beta_r x) \Big/ \prod_{r=1}^{q+i} (1-\alpha_r x),$$
*the value of* $$\{p^q\}\Delta(\alpha_1, \ldots, \alpha_{q+i})\Delta(\beta_1, \ldots, \beta_{p+i-j})$$
*is given by the determinant in Theorem XIX if the rows involving* $\beta_{p+i-j+1}, \ldots, \beta_{p+i}$ *are struck out, and all the indices of the* $\beta$*'s are decreased by* $j$*, the columns with negative indices being struck out.*

In an exactly similar manner we may obtain

XXI. *If* $$F(x) = \prod_{r=1}^{p+i} (1-\beta_r x) \Big/ \prod_{r=1}^{q+i-j} (1-\alpha_r x),$$
*then the value of*
$$\{p^q\}\Delta(\alpha_1, \ldots, \alpha_{q+i-j})\Delta(\beta_1, \ldots, \beta_{p+i})$$
*is given by the determinant in Theorem XIX if the rows involving* $\alpha_{q+i-j+1}, \ldots, \alpha_{q+i}$ *are struck out, and all the indices of the* $\alpha$*'s are decreased by* $j$*, the columns with negative indices being struck out.*

EXAMPLE. If $F(x) = (1-\beta_1 x)(1-\beta_2 x)/(1-\alpha_1 x)(1-\alpha_2 x)(1-\alpha_3 x)$, then we have
$$\{2^3\} = \prod (\alpha_r - \beta_s),$$

$$\{3^3\}\Delta(\alpha_1, \alpha_2, \alpha_3)\Delta(\beta_1, \beta_2) = \begin{array}{ccccc} \alpha_1^5, & \alpha_1^4, & \alpha_1^3, & \alpha_1^2, & \alpha_1 \\ \alpha_2^5, & \alpha_2^4, & \alpha_2^3, & \alpha_2^2, & \alpha_2 \\ \alpha_3^5, & \alpha_3^4, & \alpha_3^3, & & \alpha_3 \\ \beta_1^4, & \beta_1^3, & \beta_1^2, & \beta_1, & 1 \\ \beta_2^4, & \beta_2^3, & \beta_2^2, & \beta_2, & 1 \end{array}$$

$$\{2^2\}\Delta(\alpha_1, \alpha_2, \alpha_3)\Delta(\beta_1, \beta_2) = \begin{array}{ccccc} \alpha_1^4, & \alpha_1^3, & \alpha_1^2, & \alpha_1, & 1 \\ \alpha_2^4, & \alpha_2^3, & & \alpha_2, & 1 \\ \alpha_3^4, & \alpha_3^3, & \alpha_3^2, & \alpha_3, & 1 \\ \beta_1^3, & \beta_1^2, & \beta_1, & 1, & \\ \beta_2^3, & \beta_2^2, & \beta_2, & 1, & \end{array}$$

$$\{1^2\}\Delta(\alpha_1,\alpha_2,\alpha_3)\Delta(\beta_1,\beta_2) = \begin{matrix} \alpha_1^3, & \alpha_1^2, & \alpha_1, & 1, \\ \alpha_2^3, & \alpha_2^2, & \alpha_2, & 1, \\ \alpha_3^3, & \alpha_3^2, & \alpha_3, & 1, \\ \beta_1^3, & \beta_1^2, & \beta_1, & \\ \beta_2^3, & \beta_2^2, & \beta_2, & \end{matrix}$$

$$\{1\}\Delta(\alpha_1,\alpha_2,\alpha_3)\Delta(\beta_1,\beta_2) = \begin{matrix} \alpha_1^3, & \alpha_1^2, & \alpha_1, & 1, \\ \alpha_2^3, & \alpha_2^2, & \alpha_2, & 1, \\ \alpha_3^3, & \alpha_3^2, & \alpha_3, & 1, \\ \beta_1^2, & \beta_1, & & \\ \beta_2^2, & \beta_2, & & \end{matrix}$$

For partitions into unequal parts no corresponding simple result has been found, even for the most simple $S$-function $\{2,1\}$.

## 6.5. Relations between immanants and $S$-functions (10)

Let $[a_{st}]$ be a matrix of order $n^2$.

I. *Corresponding to any relation between S-functions of total weight n, we may replace the S-functions by the corresponding immanants of complementary coaxial minors of* $[a_{st}]$ *provided that every product is summed for all sets of complementary coaxial minors.*

It will be sufficient to prove the theorem for the product of two $S$-functions.

Let $\rho$, $\rho'$, $\rho''$ denote respectively the classes

$$(1^a, 2^b, 3^c,\ldots),\quad (1^{a'}, 2^{b'}, 3^{c'},\ldots),\quad (1^{a''}, 2^{b''}, 3^{c''},\ldots)$$

of the symmetric groups of orders $n!$, $r!$, and $s!$, for which

$$\begin{aligned} r+s &= n, \\ a'+a'' &= a, \\ b'+b'' &= b, \\ \cdot\quad \cdot\quad &\cdot\quad \cdot \end{aligned}$$

The orders of these classes are respectively $h_\rho$, $h_{\rho'}$, $h_{\rho''}$, where

$$h_\rho = n!/(1^a a!\,2^b b!\,3^c c!\ldots)$$

and the values of $h_{\rho'}$ and $h_{\rho''}$ are given by similar expressions.

Let $[a'_{st}]$ and $[a''_{st}]$ be two complementary coaxial minors of $[a_{st}]$ of orders $r^2$ and $s^2$ respectively.

Defining the product $P_S$ obtained from the matrix $[a_{st}]$ as in § 6.1, let

$$C_\rho = \sum P_S$$

denote the sum of the products $P_S$ corresponding to the $h_\rho$ permutations of the class $\rho$.

Let
$$C_{\rho'} = \sum_{\rho'} P'_S$$
denote the corresponding expression obtained from the minor $[a'_{st}]$ and the class $\rho'$, and
$$C_{\rho''} = \sum_{\rho''} P''_S$$
for the minor $[a''_{st}]$ and the class $\rho''$.

Then the product $C_{\rho'}C_{\rho''}$ will give $h_{\rho'}h_{\rho''}$ of the $h_\rho$ terms in the expression $C_\rho$.

If now we sum for the $\binom{n}{r}$ combinations of complementary coaxial minors, we shall obtain
$$\frac{n!}{r!\,s!}h_{\rho'}h_{\rho''} = \frac{n!}{r!\,s!}\,\frac{r!}{1^{a'}a'!\,2^{b'}b'!\ldots}\,\frac{s!}{1^{a''}a''!\,2^{b''}b''!\ldots} = \frac{a!}{a'!\,a''!}\,\frac{b!}{b'!\,b''!}\ldots h_\rho$$
terms which consist of the expression $C_\rho$ repeated
$$\frac{a!}{a'!\,a''!}\,\frac{b!}{b'!\,b''!}\ldots = \frac{h_{\rho'}h_{\rho''}}{h_\rho}\,\frac{n!}{r!\,s!}$$
times, corresponding to the
$$\frac{a!}{a'!\,a''!}\,\frac{b!}{b'!\,b''!}\ldots$$
ways in which the cycles of a permutation $S$ of the class $\rho$ may be separated into two separates which correspond to the classes $\rho'$ and $\rho''$ respectively.

Thus
$$\sum C_{\rho'}C_{\rho''} = \frac{h_{\rho'}h_{\rho''}}{h_\rho}\,\frac{n!}{r!\,s!}\,C_\rho. \tag{6.5; 1}$$
But since
$$S_{\rho'}S_{\rho''} = S_\rho,$$
we have
$$\left(\frac{h_{\rho'}S_{\rho'}}{r!}\right)\left(\frac{h_{\rho''}S_{\rho''}}{s!}\right) = \frac{h_{\rho'}h_{\rho''}}{h_\rho}\,\frac{n!}{r!\,s!}\left(\frac{h_\rho S_\rho}{n!}\right). \tag{6.5; 2}$$

Multiplying (6.5; 1) and (6.5; 2) by the appropriate characters and summing for all the classes, we see that if
$$\{\lambda\}\{\mu\} = \sum g_{\lambda\mu\nu}\{\nu\},$$
then
$$\sum |a'_{st}|^{(\lambda)}|a''_{st}|^{(\mu)} = \sum g_{\lambda\mu\nu}|a_{st}|^{(\nu)},$$
the summation on the left being with respect to all sets of complementary coaxial minors. This proves the theorem.

As an example illustrating this theorem, we may deduce from the equation
$$a_4 - h_1 a_3 + h_2 a_2 - h_3 a_1 + h_4 = 0$$

the known relation between corresponding determinants and permanents, viz. (**11**):†

$$\begin{pmatrix}\alpha & \beta & \gamma & \delta\\ \alpha & \beta & \gamma & \delta\end{pmatrix} - \sum \begin{pmatrix}\alpha & \beta & \gamma\\ \alpha & \beta & \gamma\end{pmatrix} \overset{+}{(}\begin{matrix}\delta\\ \delta\end{matrix}\overset{+}{)} + \sum \begin{pmatrix}\alpha & \beta\\ \alpha & \beta\end{pmatrix} \overset{+}{(}\begin{matrix}\gamma & \delta\\ \gamma & \delta\end{matrix}\overset{+}{)} -$$

$$\overset{+}{(}\begin{matrix}\beta & \gamma & \delta\\ & \gamma & \delta\end{matrix}\overset{+}{)} + \overset{+}{(}\begin{matrix}\alpha & \beta & \gamma & \delta\\ \alpha & \beta & \gamma & \delta\end{matrix}\overset{+}{)} = 0.$$

It is convenient to generalize the concept of a minor of a matrix, so as to include the cases of repetitions of rows or columns. Thus the number of $r$-rowed minors of a matrix of order $n^2$ will be

$$[n(n+1)...(n+r-1)/r!]^2$$

instead of $[n(n-1)...(n-r+1)/r!]^2$. The extra minors so introduced have each a zero determinant, and it is for this reason that they are usually ignored in determinant theory. Other immanants, however, are not necessarily equal to zero.

A conventional factor $1/r!$ is attached to every immanant of a minor for each *row* which is repeated $r$ times in the minor.

II. *Corresponding to any relation between $S$-functions we may replace each $S$-function by the corresponding immanant of a coaxial minor of $[a_{st}]$, provided that we sum with respect to all the coaxial minors of the appropriate order.*

This theorem is proved by summing Theorem I with respect to all the coaxial minors of $[a_{st}]$ of appropriate orders.

Two coaxial minors $[a'_{st}]$, $[a''_{st}]$ are complementary coaxial minors of exactly one coaxial minor $[a'''_{st}]$ of $[a_{st}]$. But if the $i$th row of $[a_{st}]$ is repeated $a_i$ times in $[a'_{st}]$ and $b_i$ times in $[a''_{st}]$, and thus $a_i+b_i$ times in $[a'''_{st}]$, then exactly $\prod[(a_i+b_i)!/a_i!\,b_i!]$ pairs of complementary coaxial minors of $[a'''_{st}]$ will coincide with $[a'_{st}]$ and $[a''_{st}]$ respectively.

Applying Theorem I to $[a'''_{st}]$, a factor $\prod[(a_i+b_i)!/a_i!\,b_i!]$ will appear in the products of the immanants of $[a'_{st}]$ and $[a''_{st}]$.

But by our convention, factors

$$1/\prod(a_i!),\quad 1/\prod(b_i!),\quad \text{and}\quad 1/\prod[(a_i+b_i)!]$$

are attached respectively to the immanants of $[a'_{st}]$, $[a''_{st}]$, and $[a'''_{st}]$. The factors will thus cancel, and by summing Theorem I for all the coaxial minors $[a'''_{st}]$ of appropriate order, the theorem is proved.

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† Muir (7), vol. iv, 459.

The following important theorem may be deduced from Theorem II.

III. *The $S$-function $\{\lambda\}$ of weight $p$ of the characteristic roots of a matrix $[a_{st}]$ is equal to the sum of the immanants corresponding to the partition $(\lambda)$ of all $p$-rowed coaxial minors of $[a_{st}]$.*

The truth of the theorem is well known for the special cases $\{\lambda\} = \{1^p\}$, since $\{1^p\} = a_p$ is the coefficient in the characteristic equation which is known to be the sum of the determinants of the $p$-rowed coaxial minors. The additional minors with repeated rows will not affect this result, since the determinants of these minors are zero.

Since every $S$-function may be expressed in terms of $a_r$'s, Theorem II enables us to deduce the theorem for the general case.

In illustration of this result, the symmetric functions $h_2$, $h_3$ of the characteristic roots of a matrix $[a_{st}]$ are given by

$$h_2 = \sum a_{ii}^2 + \sum (a_{ii}a_{jj} + a_{ij}a_{ji}),$$

$$h_3 = \sum a_{ii}^3 + \sum (a_{ii}^2 a_{jj} + 2a_{ii}a_{ij}a_{ji}) + \sum a_{ij}a_{jk}a_{ki}.$$

# VII

# $S$-FUNCTIONS OF SPECIAL SERIES

**7.1.** FOR certain special series it is possible to obtain formulae giving the value of any associated $S$-function. Such formulae may be compared with those that give the general coefficient in the Taylor series of a function, e.g. the binomial theorem. Indeed, the coefficients of the Taylor series are $S$-functions of the special type $\{n\}$, so that our problem may be regarded as a generalization.

The majority of the results considered here may be regarded as special cases of one general result, namely Theorem II which follows (1). Although other methods are available for proving the simpler results (2), it is convenient to prove the general result first and deduce these as special cases.

The first series we consider is

$$f(x) = \{(1-x)(1-qx)(1-q^2x)\dots(1-q^{N-1}x)\}^{-1}. \qquad (7.1;1)$$

To find the $S$-functions of this series we use the formula

$$\{\lambda\} = \{\lambda_1,\dots,\lambda_p\} = \frac{|\alpha_s^{\lambda_t+N-t}|}{|\alpha_s^{N-t}|}.$$

We obtain
$$\{\lambda\} = \frac{|q^{(N-s)(\lambda_t+N-t)}|}{|q^{(N-s)(N-t)}|},$$

in which $\lambda_{p+1},\dots,\lambda_N$ are taken as zero for $N > p$.

Hence, denoting $\prod (\alpha_r-\alpha_s)$ $(1 \leqslant r < s \leqslant N)$ by $\Delta(\alpha_1,\dots,\alpha_N)$ or $\Delta(\alpha_r)$, we have
$$\{\lambda\} = \frac{\Delta(q^{\lambda_r+N-r})}{\Delta(q^{N-r})}.$$

Now

$$\Delta(q^{\lambda_r+N-r}) = \prod (q^{\lambda_r+N-r}-q^{\lambda_s+N-s}) \prod (q^{\lambda_t+N-t}-q^{N-u}) \prod (q^{N-v}-q^{N-w})$$

in which

$$1 \leqslant r < s \leqslant p < v < w \leqslant N, \qquad 1 \leqslant t \leqslant p < u \leqslant N.$$

Thus

$$\frac{\Delta(q^{\lambda_r+N-r})}{\Delta(q^{N-r})}$$
$$= \prod \left(\frac{q^{\lambda_r+N-r}-q^{\lambda_s+N-s}}{q^{N-r}-q^{N-s}}\right) \prod \left(\frac{q^{\lambda_t+N-t}-q^{N-u}}{q^{N-t}-q^{N-u}}\right) \prod \left(\frac{q^{N-v}-q^{N-w}}{q^{N-v}-q^{N-w}}\right)$$

$$= \prod \left\{\frac{q^{\lambda_s}(q^{\lambda_r-\lambda_s-r+s}-1)}{(q^{s-r}-1)}\right\} \times$$

$$\times \prod \frac{(q^{\lambda_r+N-r}-q^{N-p-1})(q^{\lambda_r+N-r}-q^{N-p-2})\dots(q^{\lambda_r+N-r}-1)}{(q^{N-r}-q^{N-p-1})(q^{N-r}-q^{N-p-2})\dots(q^{N-r}-1)}$$

$$- \prod \left\{\frac{q^{\lambda_s}(q^{\lambda_r-\lambda_s-r+s}-1)}{(q^{s-r}-1)}\right. \times$$

$$\times \prod \frac{(q^{\lambda_r+p-r+1}-1)(q^{\lambda_r+p-r+2}-1)\dots(q^{\lambda_r+N-r}-1)}{(q^{p-r+1}-1)(q^{p-r+2}-1)\dots(q^{N-r}-1)}$$

$$= q^{\lambda_2+2\lambda_3+\dots} \prod \left\{\frac{(q^{\lambda_r-\lambda_s-r+s}-1)}{(q^{s-r}-1)}\right\} \times$$

$$\times \prod \frac{(q^{N-r+1}-1)(q^{N-r+2}-1)\dots(q^{N-r+\lambda_r}-1)}{(q^{p-r+1}-1)(q^{p-r+2}-1)\dots(q^{p-r+\lambda_r}-1)}$$

$$= q^{\lambda_2+2\lambda_3+\dots} \prod \left\{\frac{(1-q^{\lambda_r-\lambda_s-r+s})}{(1-q^{s-r})}\right\} \times$$

$$\times \prod \frac{(1-q^{N-r+1})(1-q^{N-r+2})\dots(1-q^{N-r+\lambda_r})}{(1-q^{p-r+1})(1-q^{p-r+2})\dots(1-q^{p-r+\lambda_r})}.$$

Hence $$\{\lambda\} = q^{\lambda_2+2\lambda_3+3\lambda_4+\dots} \prod \frac{(1-q^{\lambda_r-\lambda_s-r+s})}{(1-q^{s-r})} \frac{(P^N_\lambda)}{(P^p_\lambda)}, \tag{7.1;2}$$

where $(P^N_\lambda)$ denotes the product of the first $\lambda_i$ terms from each $i$th row of the following set of numbers:

$$\begin{matrix} 1-q^N, & 1-q^{N+1}, & 1-q^{N+2}, & \dots \\ q-q^N, & q-q^{N+1}, & q-q^{N+2}, & \dots \\ q^2-q^N, & q^2-q^{N+1}, & q^2-q^{N+2}, & \dots \end{matrix} \tag{7.1;3}$$

Again, writing $$[r]! = (1-q)(1-q^2)\dots(1-q^r), \tag{7.1;4}$$

we have $$\prod (1-q^{s-r}) = [1]!\,[2]!\dots[p\text{-}1]!,$$

so that $$\prod (1-q^{s-r})(P^p_\lambda) = \prod [r]! \prod \frac{q^{(r-1)}[\lambda_r+p-r]!}{[p-r]!}$$

$$= \prod q^{(r-1)}[\lambda_r+p-r]!.$$

Hence $$\{\lambda\} = \frac{\prod (1-q^{\lambda_r-\lambda_s-r+s})}{\prod [\lambda_r+p-r]!}(P^N_\lambda). \tag{7.1;5}$$

This proves the following result.

I. *For the series* $f(x) = [(1-x)(1-qx)\dots(1-q^{N-1}x)]^{-1}$,

$$\{\lambda_1,\dots,\lambda_p\} = q^{\lambda_2+2\lambda_3+3\lambda_4+\dots}\prod \frac{(1-q^{\lambda_r-\lambda_s-r+s})}{(1-q^{s-r})}\,\frac{(P_\lambda^N)}{(P_\lambda^p)}$$

$$= \frac{\prod (1-q^{\lambda_r-\lambda_s-r+s})}{\prod [\lambda_r+p-r]!}(P_\lambda^N),$$

*where* $$[r]! = (1-q)(1-q^2)\dots(1-q^r),$$

*and* $(P_\lambda^N)$ *is the product of the first* $\lambda_i$ *terms from each i-th row of* (7.1; 3).

**The function $\Phi(q,x)$**

Henceforward we use $\Phi(q, x)$ to denote the function of $q$ and $x$

$$\Phi(q, x) = (1-x)(1-qx)(1-q^2x)\dots(1-q^nx)\dots \text{ to } \infty. \qquad (7.1;6)$$

For $|q| < 1$, this product is absolutely convergent for all values of $x$.

The $S$-functions of $1/\Phi(q, x)$ may be obtained from Theorem I by making $N$ tend to $\infty$.

*For the series* $1/\Phi(q, x)$

$$\{\lambda_1,\dots,\lambda_p\} = q^{\lambda_2+2\lambda_3+\dots}\frac{\prod (1-q^{\lambda_r-\lambda_s-r+s})}{\prod [\lambda_r+p-r]!},$$

*where* $[r]! = (1-q)(1-q^2)\dots(1-q^r)$.

The series $f(x) = 1/(1-x)(1-qx)\dots(1-q^{N-1}x)$ of Theorem I may be expressed

$$f(x) = \Phi(q, q^Nx)/\Phi(q, x).$$

Hence Theorem I may be expressed as follows.

*If* $z = q^N$ *for some positive integral value of* $N$, *then for the series* $f(x) = \Phi(q, zx)/\Phi(q, x)$

$$\{\lambda_1,\dots,\lambda_p\} = \frac{\prod (1-q^{\lambda_r-\lambda_s-r+s})}{\prod [\lambda_r+p-r]!}(R_\lambda^z), \qquad (7.1;7)$$

*where* $(R_\lambda^z)$ *is the product of the first* $\lambda_i$ *terms from each i-th row of the set*

$$\begin{array}{lllll} 1-z, & 1-qz, & 1-q^2z, & 1-q^3z, & \dots \\ q-z, & q-qz, & q-q^2z, & q-q^3z, & \dots \\ q^2-z, & q^2-qz, & q^2-q^2z, & q^2-q^3z, & \dots \end{array} \qquad (7.1;8)$$

For this series we have

$$S_r = (1-q^{rN})/(1-q^r) = (1-z^r)/(1-q^r).$$

We now drop the restriction that $z = q^N$, taking

$$f(x) = \Phi(q,zx)/\Phi(q,x)$$

with $z$ an independent variable.

The equation $$S_r = (1-z^r)/(1-q^r)$$

still holds. Since the $S$-functions are expressible as polynomials in the $S_r$, equation (7.1; 7), which holds for $z = q^N$ for all positive integral values of $N$, must be an identity and hold without this restriction.

A more symmetric form is obtained by replacing $x$ by $wx$, which has the effect of multiplying each $S$-function $\{\lambda\}$ of weight $n$ by $w^n$, and replacing $z$ by $w^{-1}z$. We obtain

II. *For the series* $\Phi(q,zx)/\Phi(q,wx)$

$$\{\lambda_1,\ldots,\lambda_p\} = \frac{\prod (1-q^{\lambda_r-\lambda_s-r+s})}{\prod [\lambda_r+p-r]!}(R_\lambda^{wz}),$$

*where* $(R_\lambda^{wz})$ *is the product of the first* $\lambda_i$ *terms from each i-th row of the set*

$$\begin{array}{llll} w-z, & w-qz, & w-q^2z, & w-q^3z, \\ qw-z, & qw-qz, & qw-q^2z, & qw-q^3z, \\ q^2w-z, & q^2w-qz, & q^2w-q^2z, & q^2w-q^3z, \end{array} \qquad (7.1;9)$$

This is a more general result than Theorem I. By obtaining the values of $h_r$ we can express $\Phi(q,zx)/\Phi(q,wx)$ in the form

$$\Phi(q,zx)/\Phi(q,wx) = 1+\frac{(w-z)}{(1-q)}x+\frac{(w-z)(w-qz)}{(1-q)(1-q^2)}x^2+ \\ +\frac{(w-z)(w-qz)(w-q^2z)}{(1-q)(1-q^2)(1-q^3)}x^3+\ldots.$$

Some special cases are of interest. For $w = 1$, $z = -1$,

$$\frac{(1+x)(1+qx)(1+q^2x)\ldots\text{to }\infty}{(1-x)(1-qx)(1-q^2x)\ldots\text{to }\infty} \\ = 1+\frac{2x}{1-q}+\frac{2(1+q)x^2}{(1-q)(1-q^2)}+\frac{2(1+q)(1+q^2)x^3}{(1-q)(1-q^2)(1-q^3)}+\ldots$$

and $\{\lambda_1,\ldots,\lambda_p\} = \dfrac{\prod(1-q^{\lambda_r-\lambda_s-r+s})}{\prod[\lambda_r+p-r]!}$ times the product of the first $\lambda_i$ terms from each $i$th row of the set

$$\begin{matrix} 2, & 1+q, & 1+q^2, & 1+q^3, & \ldots \\ q+1, & 2q, & q+q^2, & q+q^3, & \ldots \\ q^2+1, & q^2+q, & 2q^2, & q^2+q^3, & \ldots \end{matrix} \qquad (7.1;\ 10)$$

Again, for the series

$$\Phi(q^2,qx)/\Phi(q^2,x) = \frac{(1-qx)(1-q^3x)(1-q^5x)\ldots}{(1-x)(1-q^2x)(1-q^4x)\ldots}$$

$$= 1+\frac{1-q}{1-q^2}x+\frac{(1-q)(1-q^3)}{(1-q^2)(1-q^4)}x^2+\frac{(1-q)(1-q^3)(1-q^5)}{(1-q^2)(1-q^4)(1-q^6)}x^3+\ldots,$$

writing $$[r]!! = (1-q^2)(1-q^4)\ldots(1-q^{2r}),$$

we have $\{\lambda_1,\ldots,\lambda_p\} = \dfrac{\prod(1-q^{2\lambda_r-2\lambda_s-2r+2s})}{\prod[\lambda_r+p-r]!!}$ times the product of the first $\lambda_i$ terms from each $i$th row of the set

$$\begin{matrix} 1-q, & 1-q^3, & 1-q^5, & 1-q^7, & \ldots \\ q^2-q, & q^2-q^3, & q^2-q^5, & q^2-q^7, & \ldots \\ q^4-q, & q^4-q^3, & q^4-q^5, & q^4-q^7, & \ldots \end{matrix} \qquad (7.1;\ 11)$$

These results have interpretations in the Theory of Partitions and elsewhere.

## 7.2. The functions $(1-x)^{-N}$ and $(1-x^r)^{-m}$

If in Theorem I the factors $(1-q)$, which appear equally often in numerator and denominator, are removed, and $q$ is put equal to 1, we obtain

III. *For the series* $f(x) = (1-x)^{-N}$

$$\{\lambda_1,\lambda_2,\ldots,\lambda_p\} = \frac{\prod(\lambda_r-\lambda_s-r+s)}{\prod(\lambda_r+p-r)!}(Q^N_\lambda) = \chi^{(\lambda)}_0(Q^N_\lambda)N!,$$

*where* $(Q^N_\lambda)$ *is the product of the first* $\lambda_i$ *terms from each i-th row of*

$$\begin{matrix} N, & N+1, & N+2, & \ldots \\ N-1, & N, & N+1, & \ldots \\ N-2, & N-1, & N, & \ldots \end{matrix} \qquad (7.2;\ 1)$$

**Example.** From the equation

$$\{p\}\{q\} = \{p+q\}+\{p+q-1,1\}+\{p+q-2,2\}+...+\{p,q\}$$

we may substitute the values when these are the symmetric functions of the roots of $(x-1)^m = 0$.

Remembering that

$$\chi_0^{(a,b)} = \frac{(a+b)!\,(a-b+1)}{(a+1)!\,b!}$$

and denoting the product $m(m+1)...(m+r-1)$ by $[m]_r$, we obtain

$$\frac{[m]_p[m]_q}{p!\,q!} = \frac{[m]_{p+q}}{(p+q)!}+\frac{[m]_{p+q-1}[m-1]_1(p+q-1)}{(p+q)!\,1!}+$$
$$+\frac{[m]_{p+q-2}[m-1]_2(p+q-3)}{(p+q-1)!\,2!}+\frac{[m]_{p+q-3}[m-1]_3(p+q-5)}{(p+q-2)!\,3!}+$$
$$+...(q+1) \text{ terms.}$$

In this result replace $m, p$ by $m+1, p-1$, and multiply by $m$. We obtain the equation in a slightly simpler form:

$$\frac{[m]_p[m+1]_q}{(p-1)!\,q!} = \frac{[m]_{p+q}(p+q)}{(p+q)!}+\frac{[m]_{p+q-1}[m]_1(p+q-2)}{(p+q-1)!\,1!}+$$
$$+\frac{[m]_{p+q-2}[m]_2(p+q-4)}{(p+q-2)!\,2!}+...(q+1) \text{ terms.}$$

Again, in Theorem I, if $q = -1$, and $N$ is even, say $N = 2m$, the series $f(x)$ becomes

$$f(x) = (1-x^2)^{-m}.$$

Let $p$, the number of parts in the partition $(\lambda_1,..., \lambda_p)$, be even, making $\lambda_p = 0$ if necessary. If $k$ of the $p$ terms $(\lambda_r+p-r)$ are even, then the number of even terms in $\Delta(\lambda_r+p-r)$ is

$$\tfrac{1}{2}k(k-1)+\tfrac{1}{2}(p-k)(p-k-1) = \tfrac{1}{4}[p^2-2p+(p-2k)^2].$$

This is a minimum when $k = \frac{1}{2}p$. Clearly, unless the number of odd terms $\lambda_r+p-r$ is equal to the number of even terms, the factor $(1+q)$ appears oftener in the numerator than in the denominator, and

$$\{\lambda\} = 0.$$

If the number of odd terms is equal to the number of even terms, the factors $(1+q)$ may be cancelled. The term $(1-q^r)$ becomes equal to 2 when $q = -1$, and cancels with a corresponding term in the denominator, when $r$ is odd. When $r$ is even, $(1-q^r)$ becomes $r$.

IV. *If* $f(x) = (1-x^2)^{-m}$, *and the number of even terms* $\lambda_r+p-r$ *is equal to the number of odd terms, the value of* $\{\lambda_1,\ldots,\lambda_p\}$, *for even* $p$, *is obtained by picking out the even terms only in the expression*

$$\frac{\prod(\lambda_r-\lambda_s-r+s)}{\prod(\lambda_r+p-r)!}(Q_\lambda^N).$$

*Otherwise* $$\{\lambda_1,\ldots,\lambda_p\} = 0.$$

Similarly, put $q = \cos(2\pi/r)+\sqrt{(-1)}\sin(2\pi/r)$, $N = mr$, and take the number of parts in the partition to be divisible by $r$. The series $f(x)$ becomes
$$f(x) = (1-x^r)^{-m}.$$

If the number of terms from $\lambda_r+p-r$ congruent to $j$, to modulus $r$, is $p_j$, then the number of factors in $\Delta(\lambda_r+p-r)$ divisible by $r$ is $\sum p_j(p_j-1)$, which is a minimum when

$$p_0 = p_1 = p_2 = p_3 = \ldots = p_{r-1}.$$

V. *If* $f(x) = (1-x^r)^{-m}$, *then* $\{\lambda\} = 0$, *unless*

$$p_0 = p_1 = \ldots = p_{r-1},$$

*in which case the value of* $\{\lambda\}$ *may be obtained by picking out those factors only which are divisible by r in the expression*

$$\frac{\prod(\lambda_r-\lambda_s-r+s)}{\prod(\lambda_r+p-r)!}(Q_\lambda^N).$$

Further results are immediately suggested by the above method, since $N$ need not be made a multiple of $r$.

As above, take $q = \cos(2\pi/r)+\sqrt{(-1)}\sin(2\pi/r)$. Let $N = rm+i$ $(0 \leqslant i < r)$. Take the number of parts $p$ in the partition to be congruent to $N$, to modulus $r$, making some of the parts zero if necessary. Then $f(x)$ becomes

$$f(x) = [(1-x^r)^m(1-x)(1-qx)\ldots(1-q^{i-1}x)]^{-1}.$$

If, for a partition $(\lambda)$, $p_0$, $p_1,\ldots$, $p_{r-1}$ of the terms $\lambda_j+p-j$ are congruent respectively to $0, 1,\ldots, r-1$, to modulus $r$, then $\{\lambda\}$ differs from zero only if $\sum \frac{1}{2}p_j(p_j-1)$ is a minimum. For integral values of the $p_j$ this is a minimum for any partition for which the greatest difference between the $p_j$ is unity, i.e. when $(r-i)$ of the terms $p_j$ are equal to $\alpha$ (say) and the other $i$ equal to $\alpha+1$.

VI. *If* $\{\lambda_1,\ldots,\lambda_p\}$ *is an S-function of*

$$f(x) = [(1-x^r)^m(1-x)(1-\epsilon x)\ldots(1-\epsilon^{i-1}x)]^{-1},$$

*where* $\epsilon$ *is a primitive root of* $z^r = 1$, *if* $N = rm+i$ *and* $p \equiv N \pmod r$, *and if* $p_k$ *of the terms* $\lambda_j+p-j$ *are congruent to* $k$, *to modulus* $r$, *then* $\{\lambda\} = 0$, *unless of the numbers*

$$p_0,\ p_1,\dots,\ p_{r-1}$$

*the greatest difference is unity, in which case the value of* $\{\lambda\}$ *is* $\theta$ *times the value obtained by picking out those terms only which are divisible by* $r$ *in the expression*

$$\frac{\prod(\lambda_r-\lambda_s-r+s)}{\prod(\lambda_r+p-r)!}(Q_\lambda^N),$$

*where* $\theta = \epsilon^{\lambda_2+2\lambda_3+3\lambda_4}\dots\Delta(\epsilon^{\alpha_r})/\Delta(\epsilon^{\beta_r})$, $\alpha_r$ *takes the values of the* $i$ *residues* $\lambda_j+p-j$ *which occur once oftener than the other residues, and* $\beta_r$ *takes the values* $0, 1,\dots, i-1$.

EXAMPLES. Consider the series $f(x) = 1/(1-x)(1-x^2)^m$. Here $N = 2m+1$. We find first the $S$-function $\{2^3, 1\} = \{2,2,2,1,0\}$. The zero part is added to make the number of parts congruent to $N$, to modulus 2. The numbers $\lambda_j+p-j$ take the values

$$6,\ 5,\ 4,\ 2,\ 0,$$

and the number of even terms $(p_0)$ is 4, while the number of odd terms $(p_1)$ is 1. Hence $\{\lambda\} = 0$.

We next find the value of $\{2^2, 1^2\} = \{2,2,1,1,0\}$. The numbers $\lambda_j+p-j$ take the values

$$6,\ 5,\ 3,\ 2,\ 0,$$

giving $p_0 = 3$, $p_1 = 2$, and $\{\lambda\} \neq 0$. We have

$$\frac{\prod(\lambda_r-\lambda_s-r+s)}{\prod(\lambda_r+p-r)!}(Q_\lambda^N)$$
$$= \frac{1.3.4.6.2.3.5.1.3.2.N(N+1)(N-1)N(N-2)(N-3)}{1.2.3.4.5.6.1.2.3.4.5.1.2.3.1.2}.$$

Hence, since $\theta = -1$, picking out the even terms and remembering that $N$ is odd, we get

$$\{2^2, 1^2\} = -\frac{4.6.2.2.(N+1)(N-1)(N-3)}{2.4.6.2.4.2.2}$$
$$= -\tfrac{1}{16}(N+1)(N-1)(N-3).$$

Again, take $f(x) = 1/(1-x)(1-\omega x)(1-x^3)^m$, where $\omega^3 = 1$. For the $S$-function $\{2^2, 1^2\} = \{2,2,1,1,0\}$, the numbers $\lambda_j+p-j$ take the values 6, 5, 3, 2, 0, and $p_0 = 3$, $p_1 = 0$, $p_2 = 2$. Hence $\{2^2, 1^2\} = 0$.

For the $S$-function $\{2^3, 1\} = \{2, 2, 2, 1, 0\}$, the numbers $\lambda_j+p-j$ take the values 6, 5, 4, 2, 0. Hence $p_0 = 2$, $p_1 = 1$, $p_2 = 2$, and $\{2^3, 1\} \neq 0$. We have

$$\theta = \omega^9\Delta(1, \omega^2)/\Delta(1, \omega) = 1+\omega.$$

$$\frac{\prod(\lambda_r-\lambda_s-r+s)}{\prod(\lambda_r+p-r)!}(Q_\lambda^N)$$

$$= \frac{1.2.4.6.1.3.5.2.4.2.N^2(N-1)^2(N+1)(N-2)(N-3)}{1.2.3.4.5.6.1.2.3.4.5.1.2.3.4.1.2}.$$

Picking out the terms divisible by 3, we get

$$\{2^3, 1\} = \frac{(1+\omega)6.3.(N+1)(N-2)}{3.6.3.3}$$

$$= (1+\omega)(N+1)(N-2)/9.$$

In particular it may be pointed out that the $S$-functions $\{\lambda_1,\ldots,\lambda_p\}$ corresponding to the series (i) $(1-x^3)^{-m}$, (ii) $1/(1-x)(1-x^3)^m$, (iii) $1/(1-x)(1-\omega x)(1-x^3)^m$ respectively, are multiples by integers independent of $N$ of the first $\lambda_i$ terms from each $i$th row of the following three sets:

$$\text{(i)}\quad \begin{array}{ccccccc} N, & 1, & 1, & N+3, & 1, & 1, & N+6, \\ 1, & N, & 1, & 1, & N+3, & 1, & 1, \\ 1, & 1, & N, & 1, & 1, & N+3, & 1, \\ N-3, & 1, & 1, & N, & 1, & 1, & N+3, \end{array}$$

$$\text{(ii)}\quad \begin{array}{ccccccc} 1, & 1, & N+2, & 1, & 1, & N+5, & \ldots \\ N-1, & 1, & 1, & N+2, & 1, & 1, & \ldots \\ 1, & N-1, & 1, & 1, & N+2, & 1, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

$$\text{(iii)}\quad \begin{array}{ccccccc} 1, & N+1, & 1, & 1, & N+4, & 1, & \ldots \\ 1, & 1, & N+1, & 1, & 1, & N+4, & \ldots \\ N-2, & 1, & 1, & N+1, & 1, & 1, & \ldots \\ \cdot & \cdot & \cdot & \cdot & \cdot & \cdot & \cdot \end{array}$$

Similar results hold for $S$-functions of $(1-x^r)^{-m}$, $[(1-x)(1-x^r)^m]^{-1}$, $[(1-x)(1-qx)(1-x^r)^m]^{-1}$, etc.

## 7.3. $S$-functions associated with $f(x^r)$

The similarity between the results obtained above for $(1-x)^{-m}$ and $(1-x^r)^{-m}$ suggests a possible connexion between the $S$-functions of the general series $f(x)$ and the $S$-functions of the series $f(x^r)$. It is found that the $S$-functions of the latter series are easily expressible in terms of the $S$-functions of the former.

We consider first the case $r = 2$.

Consider the symmetric functions of

$$x_1,\ x_2,\ x_3,\ \ldots,\ x_m,\ -x_1,\ -x_2,\ -x_3,\ \ldots,\ -x_m,$$

i.e. of
$$x_1,\ x_2,\ x_3,\ \ldots,\ x_{2m},$$

where
$$x_{r+m} = -x_r.$$

We have
$$\{\lambda\} = \frac{|x_s^{\lambda_t+2m-t}|}{|x_s^{2m-t}|}.$$

If in the determinant $|x_s^{\lambda_t+2m-t}|$ we replace the $r$th row by half the sum of the $r$th and the $(m+r)$th rows, and the $(m+r)$th row by half the difference, we shall obtain the same determinant save that in the first $m$ rows the terms with odd indices are replaced by zeros, and in the last $m$ rows the terms with *even* indices are replaced by zeros. If the determinant in the denominator is operated upon in the same manner, the quotient of the determinants is left unchanged. Each of these determinants may now be expressed as the product of two determinants of order $m^2$, by means of the Laplace development in terms of the first $m$ rows.

One result follows immediately. If the numbers of odd and even indices are not equal, the Laplace development shows the determinant to be zero.

*If the numbers of odd and even terms in the sequence*

$$\lambda_1+2m-1,\quad \lambda_2+2m-2,\quad \ldots,\quad \lambda_{2m}$$

*are not equal, then the $S$-function $\{\lambda_1, \lambda_2, \ldots, \lambda_{2m}\}$ of $f(x^2)$ is zero.*

If the number of odd terms is equal to the number of even terms, the $S$-function $\{\lambda\}$ may be simplified as follows. Rearrange the numbers
$$\lambda_1+2m-1,\quad \lambda_2+2m-2,\quad \ldots,\quad \lambda_{2m}$$

so that the 1st, 3rd,..., $(2m-1)$th terms are odd but remain in descending order, while the 2nd, 4th,..., $2m$th terms are even and also remain in descending order. Put $\theta = +1$ if this is a positive, and $\theta = -1$ if this is a negative permutation.

Then the Laplace developments mentioned above give

$$\{\lambda\} = \theta \frac{|x_s^{\lambda_t+2m-t}|'}{|x_s^{2m-t}|'} \frac{|x_s^{\lambda_t+2m-t}|''}{|x_s^{2m-t}|''}, \qquad (7.3;1)$$

where $s$ runs from 1 to $m$ only, and odd indices only are taken in the determinants with one accent, and even indices only are taken in the determinants with two accents.

Consider the symmetric functions of the roots of $f(x^2) = 0$, where $f(z)$ is a polynomial of degree $m$. If $S_r$ is the sum of the $r$th powers of the roots, we have

$$S_{2r+1} = 0,$$

$$S_{2r} = 2Z_r,$$

where $Z_r$ is the sum of the $r$th powers of the roots of $f(z) = 0$.

Using the result (7.3;1)

$$\{\lambda\} = \theta \frac{|x_s^{\lambda_t+2m-t}|'}{|x_s^{2m-t}|'} \frac{|x_s^{\lambda_t+2m-t}|''}{|x_s^{2m-t}|''},$$

we see that the quotients on the right may be expressed as $S$-functions of the roots of $f(z) = 0$.

Let the odd indices $\lambda_t+2m-t$ in descending order be expressed as

$$2\mu_1+2m-1, \quad 2\mu_2+2m-3, \quad \ldots, \quad 2\mu_m+1,$$

and the even indices

$$2\nu_1+2m-2, \quad 2\nu_2+2m-4, \quad \ldots, \quad 2\nu_m.$$

Then denoting the $S$-functions of the roots of $f(z) = 0$ by $\{\lambda\}'$, we have

$$\{\lambda\} = \theta\{\mu\}'\{\nu\}' = \theta \sum g_{\mu\nu\zeta}\{\zeta\}', \qquad (7.3;2)$$

where $g_{\mu\nu\zeta}$ is the coefficient of $\{\zeta\}$ in the product of $\{\mu\}$ and $\{\nu\}$.

We now consider the general case of the $S$-functions of $f(x^r)$.

The $S$-functions are given by

$$\{\lambda\} = \frac{|x_s^{\lambda_t+rm-t}|}{|x_s^{rm-t}|},$$

the determinant having $rm$ rows and columns, and $x_s$ being defined for $s > m$, by

$$x_{i+jm} = \omega^j x_i,$$

where

$$\omega = \exp(2\pi i/r).$$

The $i$th, $(i+m)$th, $(i+2m)$th,..., $(i+rm-m)$th rows of both determinants are replaced respectively by

$$\frac{1}{r}[i\text{th row}+(i+m)\text{th row}+(i+2m)\text{th row}+...],$$

$$\frac{1}{r}[i\text{th row}+\omega^{-1}(i+m)\text{th row}+\omega^{-2}(i+2m)\text{th row}+...],$$

$$\frac{1}{r}[i\text{th row}+\omega^{-2}(i+m)\text{th row}+\omega^{-4}(i+2m)\text{th row}+...],$$

thus picking out separately the indices which are congruent to 0, 1, 2,..., $r-1$ to modulus $r$.

If the numbers of terms of the sequence

$$\lambda_1+rm-1, \quad \lambda_2+rm-2, \quad \lambda_3+rm-3, \quad ..., \quad \lambda_{rm}$$

congruent respectively to 0, 1, 2,..., $(r-1)$, $(\text{mod}\, r)$, are not all equal, then

$$\{\lambda\} = 0.$$

If the numbers of these terms are equal, we have

$$\{\lambda\} = \theta \prod \frac{|x_s^{\lambda_t+rm-t}|'}{|x_s^{rm-t}|'}, \tag{7.3; 3}$$

in which the determinants have $m$ rows and columns, and only those indices are picked out which have the same residue to modulus $r$; $\theta$ is $+1$ or $-1$ according as the rearrangement of the sequence $\lambda_i+rm-i$, such that the $(ar+i)$th term is congruent to $r-i$, to modulus $r$, but the terms congruent to modulus $r$ are still in descending order, is a positive or a negative permutation. The product is taken for all residues of the indices to modulus $r$.

VII. *If the numbers of the sequence*

$$\lambda_1+rm-1, \quad \lambda_2+rm-2, \quad \lambda_3+rm-3, \quad ..., \quad \lambda_{rm}$$

*congruent respectively to* 0, 1, 2,..., $r-1$ $(\text{mod}\, r)$ *are not all equal,* $\{\lambda\} = 0$. *If they are equal and those congruent to* $q\,(\text{mod}\, r)$ *are*

$$r[\mu_{q1}+m-1]+q, \quad r[\mu_{q2}+m-2]+q, \quad ..., \quad r\mu_{qm}+q,$$

*then*

$$\{\lambda\} = \theta\{\mu_{01}, \mu_{02},..., \mu_{0m}\}'\{\mu_{11},..., \mu_{1m}\}'...\{\mu_{r-1,1},..., \mu_{r-1,m}\}',$$

*where* $\{\lambda\}$ *denotes an S-function of* $f(x^r)$ *and* $\{\mu\}'$ *an S-function of* $f(x)$.

The expressions for the $S$-functions of $1/[(1-x)(1-x^r)^m]$ found above suggest a modification of this result for the $S$-functions of

$f(x^r)$. We now consider the $S$-functions of $(1-\zeta x)^{-1}f(x^r)$. In the first place let

$$f(x) = [(1-\zeta x)(1-\alpha_1^2 x^2)(1-\alpha_2^2 x^2)...(1-\alpha_r^2 x^2)]^{-1}.$$

For this series we obtain for the general $S$-function, putting $N = 2r+1$,

$$\{\lambda_1,..., \lambda_p\} = \begin{vmatrix} \alpha_s^{\lambda_t+N-t} \\ \hdashline (-\alpha_s)^{\lambda_t+N-t} \\ \hdashline \zeta^{\lambda_t+N-t} \end{vmatrix} \div \begin{vmatrix} \alpha_s^{N-t} \\ \hdashline (-\alpha_s)^{N-t} \\ \hdashline \zeta^{N-t} \end{vmatrix}$$

Operate on both determinants by taking half the sum of the $i$th and the $(r+i)$th rows to replace the $i$th row, and half the difference to replace the $(r+i)$th row, for $1 \leqslant i \leqslant r$. The effect is to pick out the even and the odd indices respectively in the first and the second set of $r$ rows. The last row is unaltered.

Three cases arise. Amongst the $N = 2r+1$ terms $\lambda_i+N-i$ there may be (i) $r+1$ even and $r$ odd terms, (ii) $r$ even and $r+1$ odd terms, or (iii) the difference between the numbers of odd and even terms may be greater than 1.

In case (iii) the Laplace development of the first determinant in terms of the first $r$ rows gives immediately

$$\{\lambda\} = 0.$$

In case (i) we obtain

$$\{\lambda_1,..., \lambda_p\} = \begin{vmatrix} \alpha_s^{\lambda_t+N-t} \\ \hdashline \zeta^{\lambda_t+N-t} \end{vmatrix}' \left|\alpha_s^{\lambda_t+N-t}\right|'' \div \begin{vmatrix} \alpha_s^{N-t} \\ \hdashline \zeta^{N-t} \end{vmatrix} \left|\alpha_s^{N-t}\right|'$$

a single or a double accent against a determinant denoting respectively that even indices only or odd indices only are taken.

Denote $S$-functions of

$$[(1-\alpha_1^2 x)(1-\alpha_2^2 x)...]^{-1} \quad \text{and} \quad [(1-\zeta x)(1-\alpha_1^2 x)(1-\alpha_2^2 x)...]^{-1}$$

by $\{\lambda; \alpha^2\}$ and $\{\lambda; \alpha^2, \zeta\}$ respectively. Then, if the even terms $\lambda_r+N-r$ in decreasing order are

$$2\mu_1+2r, \quad 2\mu_2+2r-2, \quad ..., \quad 2\mu_{r+1},$$

and the odd terms

$$2\nu_1+2r-1, \quad 2\nu_2+2r-3, \quad ..., \quad 2\nu_r+1,$$

we obtain $$\{\lambda\} = \theta\{\mu_1,..., \mu_{r+1}; \alpha^2, \zeta\}\{\nu_1,..., \nu_r; \alpha^2\}$$

where $\theta$ is $+1$ or $-1$ according as

$$2\mu_1+2r, \quad 2\nu_1+2r-1, \quad 2\mu_2+2r-2, \quad ..., \quad 2\nu_r+1, \quad 2\mu_{r+1}$$

is an even or odd permutation of

$$\lambda_1+N-1, \quad \lambda_2+N-2, \quad ..., \quad \lambda_N.$$

In case (ii) the powers of $\zeta$ go with the determinant with odd indices. If the odd terms $\lambda_r+N-r$ are

$$2\mu_1+2r+1,\quad 2\mu_2+2r-1,\quad \ldots,\quad 2\mu_{r+1}+1,$$

and the even indices

$$2\nu_1+2r-2,\quad 2\nu_2+2r-4,\quad \ldots,\quad 2\nu_r,$$

then, as before,

$$\{\lambda\} = \theta\{\mu_1,\ldots,\mu_{r+1};\, \alpha^2, \zeta\}\{\nu_1,\ldots,\nu_r;\, \alpha^2\},$$

where $\theta$ is $+1$ or $-1$ according as

$$2\nu_1+2r-2,\ 2\mu_1+2r+1,\ 2\nu_2+2r-4,\ 2\mu_2+2r-1,\ldots,\ 2\mu_r+3,\ 2\mu_{r+1}+1$$

is an even or an odd permutation of

$$\lambda_1+N-1,\quad \lambda_2+N-2,\quad \ldots,\quad \lambda_N.$$

In the calculation we may take the smallest value of $r$ such that the number of parts in the partition $(\lambda)$ does not exceed $2r+1$. Larger values of $r$ merely add zero parts to the partitions $(\mu)$ and $(\nu)$. The generalization to the $S$-functions of any series follows the proof for other properties of $S$-functions.

VIII. *Let the S-functions respectively of* $(1-\zeta x)^{-1}f(x^2)$, $(1-\zeta^2 x)^{-1}f(x)$, *and* $f(x)$ *be denoted by* $\{\lambda;\, (2)\zeta\}$, $\{\lambda;\, (1)\zeta\}$, *and* $\{\lambda;\, (1)\}$, *and let* $(\lambda_1,\ldots,\lambda_{2p+1})$ *be any partition for which* $\lambda_{2p} \neq 0$. *If the difference between the numbers of odd and even terms in*

(i) $$\lambda_1+2p,\quad \lambda_2+2p-1,\quad \ldots,\quad \lambda_{2p+1}$$

*exceeds* 1, *then* $$\{\lambda_1,\ldots,\lambda_{2p+1};\, (2)\zeta\} = 0.$$

*Otherwise, if the series* (i) *contains* $p+1$ *even and* $p$ *odd terms, let these be*

(ii) $2\mu_1+2p,\quad 2\nu_1+2p-1,\quad 2\mu_2+2p-2,\quad \ldots,\quad 2\nu_p+1,\quad 2\mu_{p+1},$

*where* $\mu_1 \geqslant \mu_2 \geqslant \ldots \geqslant \mu_{p+1}$ *and* $\nu_1 \geqslant \nu_2 \geqslant \ldots \geqslant \nu_p$.

*If there are* $p$ *even and* $p+1$ *odd terms in* (i), *let them be*

(iii) $2\nu_1+2p-2,\ 2\mu_1+2p+1,\ 2\nu_2+2p-4,\ 2\mu_2+2p-1,\ldots,\ 2\nu_p,\ 2\mu_p+3,\ 2\mu_{p+1}+1.$

*In either case*

$$\{\lambda_1,\ldots,\lambda_{2p+1};\, (2)\zeta\} = \theta\{\mu_1,\ldots,\mu_{p+1};\, (1)\zeta\}\{\nu_1,\ldots,\nu_p;\, (1)\},$$

*where* $\theta$ *is* $+\zeta^j$ *or* $-\zeta^j$ *according as* (ii) *or* (iii) *is an even or odd permutation of* (i), *and* $j = 0$ *in case* (ii), $j = 1$ *in case* (iii).

A similar result for $(1-\zeta x)^{-1}f(x^r)$, proved in the same way, is as follows:

IX. *Let* $\{\lambda; (r)\zeta\}$, $\{\lambda; (1)\zeta\}$, *and* $\{\lambda; (1)\}$ *denote S-functions respectively of* $(1-\zeta x)^{-1}f(x^r)$, $(1-\zeta^r x)^{-1}f(x)$, *and* $f(x)$, *and let* $(\lambda_1,\dots,\lambda_{rp+1})$ *be a partition such that* $\lambda_{rp-r+2} \neq 0$. *If the sequence*

$$\text{(i)} \qquad \lambda_1+rp, \quad \lambda_2+rp-1, \quad \dots, \quad \lambda_{rp+1}$$

*is such that the number of terms congruent to* $i$, *to modulus* $r$, *is less than* $p$ *for any* $i$, *then*

$$\{\lambda_1,\dots,\lambda_{rp+1}; (r)\zeta\} = 0.$$

*Otherwise let* $j$ *be that number such that there are* $p+1$ *terms in* (i) *congruent to* $j$, *and let the terms in* (i) *be arranged as follows:*

$$\begin{array}{llll}
\text{(ii)}\ r\mu_{r-1,1}+r(p-1)+r-1, & r\mu_{r-1,2}+r(p-2)+r-1, & \dots, & r\mu_{r-1,p}+r-1, \\
\quad r\mu_{r-2}\zeta_1+r(p-1)+r-2, & & \dots, & \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
\quad r\mu_{j,1}+rp+j, & r\mu_{j,2}+r(p-1)+j, & \dots, & r\mu_{j,p}+r+j, \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
\cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot\ \cdot\ \cdot & \cdot\ \cdot & \cdot\ \cdot\ \cdot\ \cdot \\
\quad r\mu_{0,1}+r(p-1), & r\mu_{0,2}+r(p-2), & \dots, & r\mu_{0,p},
\end{array}$$

*and* $$r\mu_{j,p+1}+j,$$

*for which* $$\mu_{i,1} \geqslant \mu_{i,2} \geqslant \dots \geqslant \mu_{i,p},$$
$$\mu_{j,1} \geqslant \mu_{j,2} \geqslant \dots \geqslant \mu_{j,p} \geqslant \mu_{j,p+1}.$$

*Then*

$$\{\lambda_1,\dots,\lambda_{rp+1}; (r)\zeta\} = \theta\{\mu_{j,1},\dots,\mu_{j,p+1}; (1)\zeta\} \prod_{i\neq j} \{\mu_{i,1},\dots,\mu_{i,p}; (1)\}$$

*where* $\theta = +\zeta^j$ *or* $-\zeta^j$ *according as* (ii) *is an even or an odd permutation of* (i).

# VIII

# THE CALCULATION OF THE CHARACTERS OF THE SYMMETRIC GROUP

## 8.1. Frobenius's formula

THE actual numerical values of the characters of groups, especially the symmetric groups, are of such importance that we devote this chapter to methods for the computation of the characters of the symmetric groups. At the end of the book will be found tables of these characters up to and including those for the symmetric group of order 10!.

Of the methods of calculation, precedence must be given to the use of Frobenius's formula (5.2; 8) (1):

$$S_\rho \Delta(x_1,\dots,x_m) = \sum \pm \chi_\rho^{(\lambda)} x_1^{\lambda_1+m-1} x_2^{\lambda_2+m-2} \dots x_m^{\lambda_m}.$$

This formula is theoretically sufficient for all cases; and indeed, for theory, it has a very remarkable simplicity. For numerical computation, however, the large number of terms involved make its use unwieldy except when $m$ is small, and much quicker methods are available. However, the formula readily gives formulae for characters corresponding to partitions into two or even three parts.

Consider the character $\chi^{(n-q,q)}$ of the symmetric group of order $n!$. For the class $\rho = (1^a\, 2^b\, 3^c \dots)$, $\chi_\rho^{(n-q,q)}$ is the coefficient of $x_1^{n-q+1} x_2^q$ in the product

$$(x_1+x_2)^a (x_1^2+x_2^2)^b (x_1^3+x_2^3)^c \dots (x_1-x_2),$$

which is $$\chi_\rho^{(n-q,q)} = \sum \binom{a}{l}\binom{b}{m}\binom{c}{n} - \sum \binom{a}{l'}\binom{b}{m'}\binom{c}{n'},$$

the summations being with respect to all solutions of

$$l+2m+3n+\dots = q,$$
$$l'+2m'+3n'+\dots = q-1.$$

Hence, for example, it is easily verified that

$$\chi_\rho^{(n-1,1)} = a-1,$$
$$\chi_\rho^{(n-2,2)} = \tfrac{1}{2}a(a-3)+b,$$
$$\chi_\rho^{(n-3,3)} = \tfrac{1}{6}a(a-1)(a-5)+b(a-1)+c,$$

It should be remembered that (§ 5.3, Theorems IV, V) $\chi_\rho^{(1^n)} = \pm 1$ according as $\rho$ contains positive or negative permutations, and also that if $(\lambda) = (\tilde{\mu})$,

$$\chi_\rho^{(\lambda)} = \chi_\rho^{(\mu)}\,\chi_\rho^{(1^n)}.$$

Hence we are enabled to deduce from these equations the values of the characters $\chi^{(2,1^{n-2})}$, $\chi^{(2^2,1^{n-4})}$, $\chi^{(2^3,1^{n-6})}$,.... Indeed, it is only necessary to compute one half of a table of characters, and from each character the values of the character corresponding to the conjugate partition may be written down.

### S-functions of special series (2)

The $S$-functions of special series yield many relations between the characters of the symmetric groups. In particular we refer to the equation

$$\prod \frac{(1-z^r)^{a_r}}{(1-q^r)^{a_r}} = \sum \chi_\rho^{(\lambda)} \frac{\prod (1-q^{\lambda_i-\lambda_j-i+j})}{\prod [\lambda_i+p-i]!} (R_\lambda^{w,z}),$$

where $\rho$ denotes the class $(1^{a_1}2^{a_2}3^{a_3}\ldots)$ and $(R_\lambda^{w,z})$ is the product of the first $\lambda_i$ terms from each $i$th row of

$$\begin{array}{llll} w-z, & w-qz, & w-q^2z, & \ldots, \\ qw-z, & qw-qz, & qw-q^2z, & \ldots, \end{array}$$

This equation is obtained from the equation

$$S_\rho = \sum \chi_\rho^{(\lambda)}\{\lambda\}$$

by substituting from Theorem II, § 7.1.

This equation is sufficient to determine all the characters of the symmetric groups, although in this form it is not easy to handle. By the following method we may express the values of any character in terms of the cycles in the classes.

Take $w = 1$. Corresponding to any partition $(\lambda)$, in the expression $(R_\lambda^{w,z})$ the least power of $q$ that occurs has index $\lambda_2+2\lambda_3+3\lambda_4+\ldots$. Hence if we pick out the coefficient of $q^r$, only those characteristics occur for which the partition $(\lambda)$ satisfies

$$\lambda_2+2\lambda_3+3\lambda_4+\ldots \leqslant r.$$

From the coefficients of $1, z, z^2, z^3,\ldots$ we obtain

$$\begin{aligned}
1 &= \chi_\rho^{(n)}, \\
a_1 &= \chi_\rho^{(n)}+\chi_\rho^{(n-1,1)}, \\
\tfrac{1}{2}a_1(a_1-1)-a_2 &= \chi_\rho^{(n-1,1)}+\chi_\rho^{(n-2,1^2)}, \\
\tfrac{1}{6}a_1(a_1-1)(a_1-2)-a_1a_2+a_3 &= \chi^{(n-2,1^2)}+\chi^{(n-3,1^3)}, \\
\tfrac{1}{24}a_1(a_1-1)(a_1-2)(a_1-3)-\tfrac{1}{2}a_1(a_1-1)a_2+a_1a_3- & \\
-a_4+\tfrac{1}{2}a_2(a_2-1) &= \chi^{(n-3,1^3)}+\chi^{(n-4,1^4)},
\end{aligned}$$

. . . . . . . . . . . . . . . . . . . .

whence by solving these equations

$$\begin{aligned}
\chi^{(n-1,1)} &= a_1-1, \\
\chi^{(n-2,1^2)} &= \tfrac{1}{2}(a_1-1)(a_1-2)-a_2, \\
\chi^{(n-3,1^3)} &= \tfrac{1}{6}(a_1-1)(a_1-2)(a_1-3)-(a_1-1)a_2+a_3, \\
\chi^{(n-4,1^4)} &= \tfrac{1}{24}(a_1-1)(a_1-2)(a_1-3)(a_1-4)-\tfrac{1}{2}(a_1-1)(a_1-2)a_2+ \\
&\qquad +(a_1-1)a_3-a_4+\tfrac{1}{2}a_2(a_2-1),
\end{aligned}$$

. . . . . . . . . . . . . . . . . . . .

From the coefficients of $qz, qz^2, qz^3,\ldots$ we obtain

$$\begin{aligned}
a_1^2 &= \chi_\rho^{(n-2,1^2)}+\chi_\rho^{(n-2,2)}+3\chi_\rho^{(n-1,1)}+2\chi_\rho^{(n)}, \\
\tfrac{1}{2}a_1^2(a_1-1)-a_1a_2 &= \chi_\rho^{(n-3,1^3)}+\chi_\rho^{(n-3,2,1)}+2\chi_\rho^{(n-2,2)}+3\chi_\rho^{(n-2,1^2)}+ \\
&\qquad +3\chi_\rho^{(n-1,1)}+\chi_\rho^{(n)}, \\
\tfrac{1}{6}a_1^2(a_1-1)(a_1-2)-a_1^2a_2+a_1a_3 & \\
&= \chi^{(n-4,1^4)}+\chi^{(n-4,2,1^2)}+3\chi^{(n-3,1^3)}+\chi^{(n-2,2)}+2\chi^{(n-3,2,1)},
\end{aligned}$$

. . . . . . . . . . . . . . . . . . . .

whence

$$\begin{aligned}
\chi^{(n-2,2)} &= \tfrac{1}{2}a_1(a_1-3)+a_2, \\
\chi^{(n-3,2,1)} &= \tfrac{1}{3}a_1(a_1-2)(a_1-4)-a_3, \\
\chi^{(n-4,2,1^2)} &= \tfrac{1}{8}a_1(a_1-2)(a_1-3)(a_1-5)-\tfrac{1}{2}a_2a_1(a_1-3)-\tfrac{1}{2}a_2(a_2-1)+a_4,
\end{aligned}$$

. . . . . . . . . . . . . . . . . . . .

Similarly, from the coefficients of $q^2z, q^2z^2, q^2z^3,\ldots$ we may obtain

$$\begin{aligned}
\chi^{(n-3,3)} &= \tfrac{1}{6}a_1(a_1-1)(a_1-5)+a_2(a_1-1)+a_3, \\
\chi^{(n-4,3,1)} &= \tfrac{1}{8}a_1(a_1-1)(a_1-3)(a_1-6)+\tfrac{1}{2}a_1a_2(a_1-3)-\tfrac{1}{2}a_2(a_2-3)-a_4,
\end{aligned}$$

### Recurrence relations

For the actual calculation of the tables of characters of the symmetric groups, by far the quickest method is by the use of recurrence relations.

Let $\rho$ be a class of the symmetric group of order $n!$, and $\rho'$ the class of the symmetric group of order $(n+1)!$ with the same cycles as the class $\rho$, and one extra cycle of order 1, so that

$$S_{\rho'} = S_1 S_\rho.$$

Now
$$S_\rho = \sum \chi_\rho^{(\lambda)}\{\lambda\},$$
$$S_{\rho'} = \sum \chi_{\rho'}^{(\mu)}\{\mu\},$$
$$S_1 = \{1\}.$$

Hence
$$\sum \chi_{\rho'}^{(\mu)}\{\mu\} = \sum \chi_\rho^{(\lambda)}\{\lambda\}\{1\}.$$

Equating coefficients of $\{\mu\}$ we have

$$\chi_{\rho'}^{(\mu)} = \sum \chi_\rho^{(\lambda)}$$

summed for all partitions $(\lambda)$ such that $\{\mu\}$ appears in the product $\{\lambda\}\{1\}$ (3).

If $(\mu) = (\mu_1, \mu_2,..., \mu_k)$, the corresponding partitions $(\lambda)$ are

$$(\mu_1-1, \mu_2,..., \mu_k), \quad (\mu_1, \mu_2-1,..., \mu_k), \quad ..., \quad (\mu_1, \mu_2,..., \mu_k-1),$$

those terms being omitted for which the descending order of the parts is destroyed.

This formula enables the characteristics of those classes of the symmetric group of order $(n+1)!$, which have at least one cycle of order 1, to be written down from the table of characters of the symmetric group of order $n!$.

Thus, given the following table of characters of the symmetric group of order 3!:

| Class | $1^3$ | $1\,2$ | $3$ |
|---|---|---|---|
| Order | 1 | 3 | 2 |
| $\chi^{(3)}$ | 1 | 1 | 1 |
| $\chi^{(21)}$ | 2 | 0 | $-1$ |
| $\chi^{(1^3)}$ | 1 | $-1$ | 1 |

from the equations

$$\chi_{\rho'}^{(4)} \qquad \chi_\rho^{(3)},$$
$$\chi_{\rho'}^{(31)} = \chi_\rho^{(3)} + \chi_\rho^{(21)},$$
$$\chi_{\rho'}^{(2^2)} = \chi_\rho^{(21)},$$

we may obtain the following part of the table of characters of the symmetric group of order 4!.

| Class | $1^4$ | $1^2 2$ | $1\,3$ | $4$ | $2^2$ |
|---|---|---|---|---|---|
| Order | 1 | 6 | 8 | 6 | 3 |
| $\chi^{(4)}$ | 1 | 1 | 1 | | |
| $\chi^{(31)}$ | 3 | 1 | 0 | | |
| $\chi^{(2^2)}$ | 2 | 0 | $-1$ | | |
| $\chi^{(21^2)}$ | 3 | $-1$ | 0 | | |
| $\chi^{(1^4)}$ | 1 | $-1$ | 1 | | |

The theorem may be generalized to the case of one cycle of order $r$, and this allows the whole table of characters of any symmetric group to be completed from the tables of characters of lower degrees.

Let $\rho$ be a class of the symmetric group of order $n!$, and $\rho'$ the class of the symmetric group of order $(n+r)!$ which has the same cycles as the class $\rho$ together with an extra cycle of order $r$. Thus

$$S_{\rho'} = S_r S_\rho.$$

From Frobenius's formula

$$S_\rho \Delta(x_1,\dots, x_m) = \sum \pm\chi_\rho^{(\lambda)}\, x_1^{\lambda_1+m-1}\dots x_m^{\lambda_m},$$

$$S_r S_\rho \Delta(x_1,\dots, x_m) = \sum \pm\chi_{\rho'}^{(\mu)}\, x_1^{\mu_1+m-1}\dots x_m^{\mu_m}.$$

Remembering that $S_r = \sum_i x_i^r$, by multiplying the first equation by $S_r$ and comparing with the second, it is clear that

$$\chi_{\rho'}^{(\mu)} = \sum \pm\chi_\rho^{(\lambda)},$$

summed for the partitions $(\lambda)$ such that the sequence

$$\lambda_1+m-1,\quad \lambda_2+m-2,\quad \dots,\quad \lambda_m$$

is obtained from the sequence

$$\mu_1+m-1,\quad \mu_2+m-2,\quad \dots,\quad \mu_m$$

by decreasing one term by $r$ and rearranging in descending order. The minus sign is taken if the rearrangement corresponds to a negative permutation (4).

Referring now to our convention (§ 6.4) for $S$-functions when the parts are expressed in non-descending order, it is clear that the characters $\chi_\rho^{(\lambda)}$ are those which correspond to the $S$-functions

$$\{\mu_1-r, \mu_2,\dots, \mu_m\},\quad \{\mu_1, \mu_2-r,\dots, \mu_m\},\quad \dots,\quad \{\mu_1, \mu_2,\dots, \mu_m-r\}.$$

THEORÊM. *If the class* $\rho'$ *of the symmetric group of order* $(n+r)!$ *contains the same cycles as the class* $\rho$ *of the symmetric group of order* $n!$, *together with an extra cycle of order* $r$, *then*

$$\chi_{\rho'}^{(\mu)} = \sum \pm \chi_{\rho}^{(\lambda)},$$

*summed for the characters* $\chi^{(\lambda)}$ *which correspond to the S-functions*

$$\{\mu_1-r, \mu_2,\ldots, \mu_m\}, \quad \{\mu_1, \mu_2-r,\ldots, \mu_m\}, \quad \ldots, \quad \{\mu_1, \mu_2,\ldots, \mu_m-r\},$$

*the minus sign being taken when the S-function, with the parts reduced to descending order, becomes negative.*

As an example we find the values taken by the character $\chi^{(4\,3^2\,1^2)}$ of the symmetric group of order 12!, for those classes which contain a cycle of order 5.

The corresponding $S$-functions are

$$\{-1,3,3,1,1\}, \quad \{4,-2,3,1,1\}, \quad \{4,3,-2,1,1\}, \quad \{4,3,3,-4,1\},$$
$$\{4,3,3,1,-4\}$$

which reduce to

$$+\{2,2,1,1,1\}, \quad +\{4,2,0,0,1\}, \quad +\{4,3,0,0,0\}, \quad -\{4,3,3,2,-3\},$$
$$+\{4,3,3,1,-4\}.$$

The second, fourth, and fifth of these are zero. Hence

$$\chi_{\rho'}^{(4\,3^2\,1^2)} = \chi_{\rho}^{(4\,3)} + \chi_{\rho}^{(2^2\,1^3)}.$$

For the classes

$1^7$, $1^5 2$, $1^4 3$, $1^3 4$, $1^3 2^2$, $1^2 2\, 3$, $1^2 5$, $1\, 6$, $1\, 2\, 4$, $1\, 2^3$, $1\, 3^2$, $2\, 5$, $2^2 3$, $3\, 4$, $7$

the characters $\chi^{(4\,3)}$ and $\chi^{(2^2\,1^3)}$ take the values

14, 4, −1, −2, 2, 1, −1, 0, 0, 0, 2, −1, −1, 1, 0,
14, −6, 2, 0, 2, 0, −1, 1, 0, −2, −1, 1, 2, 0, 0.

Hence $\chi^{(4\,3^2\,1^2)}$ for the given classes takes the following values

| $1^7 5$, | $1^5 2\, 5$, | $1^4 3\, 5$, | $1^3 4\, 5$, | $1^3 2^2 5$, | $1^2 2\, 3\, 5$, | $1^2 5^2$, | $1\, 5\, 6$, |
|---|---|---|---|---|---|---|---|
| 28, | −2, | 1, | −2, | 4, | 1, | −2, | 1, |

| $1\, 2\, 4\, 5$, | $1\, 2^3 5$, | $1\, 3^2 5$, | $2\, 5^2$, | $2^2 3\, 5$, | $3\, 4\, 5$, | $5\, 7$, |
|---|---|---|---|---|---|---|
| 0, | −2, | 1, | 0, | 1, | 1, | 0. |

**Congruences (5)**

The preceding method is the most practical way of computing the tables of characters of the symmetric groups. We take this opportunity, however, of mentioning some other properties of the characters which may be used for completing a table when the characteristics of some classes are known.

If $p$ is a prime,

$$(x+y)^p \equiv x^p+y^p \pmod{p},$$

and it follows that

$$S_1^p \equiv S_p \pmod{p}.$$

In Frobenius's formula for the characteristics of a given class $\rho$ with at least $p$ cycles of order 1, replace $S_1^p$ by $S_p$. The result is congruent to the original expression to modulus $p$, and is Frobenius's formula for the class in which the $p$ cycles of order 1 are replaced by one cycle of order $p$.

Hence, *if two classes $\rho$, $\rho'$ have the same cycles save that $p$ cycles of order* 1 *in $\rho$ are replaced by a cycle of order $p$ in $\rho'$, the characteristics of the two classes are congruent to modulus $p$.*

More generally, *corresponding to any congruence between expressions of the form $(x^a+y^a)^b$ regarded as functions of $x$ and $y$, there is a corresponding congruence between the characteristics of the symmetric group.*

Thus, since $(x^a+y^a)^p \equiv (x+y)^{ap} \pmod{p}$ when $p$ is prime, $p$ cycles of order $a$ may be replaced by a cycle of order $ap$, and the characteristics are congruent to modulus $p$.

Again, $(x^2+y^2)^4 \equiv (x+y)^8 \pmod{8}$, and the characteristics of the two classes $(1^8)$ and $(2^4)$ of the symmetric group of order 8!, are congruent to modulus 8.

### Classes for which the orders of the cycles have a common factor (6)

Suppose first that the common factor is 2. The following method enables us to express the characteristics of a class with only even cycles in terms of the characters of a symmetric group of lower order.

Consider the $S$-functions of the roots of $f(x^2) = 0$, where $f(z)$ is a polynomial of degree $m$. If $S_r$ is the sum of the $r$th powers of the roots of $f(x^2) = 0$, and $Z_r$ the sum of the $r$th powers of the roots of $f(z) = 0$, we have

$$S_{2r+1} = 0,$$

$$S_{2r} = 2Z_r.$$

Denote by $\{\lambda\}$, $\{\lambda\}'$ respectively the $S$-functions of the roots of $f(x^2) = 0$ and $f(z) = 0$. We use the result (§ 7.3, Theorem VII),

$$\{\lambda\} \equiv \{\lambda_1, \lambda_2,\dots, \lambda_{2m}\} = 0$$

unless the numbers of even and odd terms are equal in the sequence

$$\lambda_1+2m-1,\quad \lambda_2+2m-2,\quad \ldots,\quad \lambda_{2m},$$

in which case if the odd terms be expressed in descending order as

$$2\mu_1+2m-1,\quad 2\mu_2+2m-3,\quad \ldots,\quad 2\mu_m+1,$$

and the even terms as

$$2\nu_1+2m-2,\quad 2\nu_2+2m-4,\quad \ldots,\quad 2\nu_m;$$

then

$$\{\lambda\} = \theta\{\mu\}'\{\nu\}' = \theta \sum g_{\mu\nu\zeta}\{\zeta\}',$$

where $\theta$ is $\pm 1$ according as

$$2\mu_1+2m-1,\quad 2\nu_1+2m-2,\quad 2\mu_2+2m-3,\quad \ldots,\quad 2\nu_m$$

is a positive or negative permutation of

$$\lambda_1+2m-1,\quad \lambda_2+2m-2,\quad \lambda_3+2m-3,\quad \ldots,\quad \lambda_{2m}.$$

Denote by $\rho$ the class $(1^{\alpha_1}2^{\alpha_2}3^{\alpha_3}\ldots)$ of the symmetric group of order $m!$, and by $\rho'$ the class $(2^{\alpha_1}4^{\alpha_2}6^{\alpha_3}\ldots)$ of the symmetric group of order $2m!$. Then

$$h_\rho = m!/1^{\alpha_1}2^{\alpha_2}\ldots(\alpha_1!\,\alpha_2!\ldots),$$

$$h_{\rho'} = (2m)!/2^{\alpha_1}4^{\alpha_2}\ldots(\alpha_1!\,\alpha_2!\ldots),$$

$$(2m)!h_\rho = 2^p m!\,h_{\rho'},$$

where

$$p = \alpha_1+\alpha_2+\alpha_3+\ldots,$$

so that we have

$$Z_\rho = 2^p S_{\rho'}.$$

Thus

$$\{\lambda\} = \frac{1}{(2m)!}\sum \chi_{\rho'}^{(\lambda)} h_{\rho'} S_{\rho'}$$

$$= \frac{1}{m!}\sum \chi_{\rho'}^{(\lambda)} h_\rho Z_\rho$$

$$= \frac{\theta}{m!}\sum g_{\mu\nu\zeta} h_\rho \chi_\rho^{(\zeta)} Z_\rho.$$

Hence

$$\chi_{\rho'}^{(\lambda)} = \theta \sum g_{\mu\nu\zeta}\chi_\rho^{(\zeta)}.$$

This is the required result. As an example we find the values of the character $\chi^{(4\,3^2\,2)}$ of the symmetric group of order 12! for all classes with cycles of even order only.

Let $(\lambda)$ denote the partition

$$12 = 4+3+3+2.$$

Taking $2m = 4$, the sequence $\lambda_i+2m-i$ becomes

$$7,\quad 5,\quad 4,\quad 2.$$

The odd terms are $2.2+3,\quad 2.2+1,$

and the even terms $2.1+2,\quad 2.1.$

Hence

$$\{\mu\} = \{2^2\}, \qquad \{\nu\} = \{1^2\},$$
$$\{\mu\}\{\nu\} = \{3^2\}+\{3\,2\,1\}+\{2^2\,1^2\}.$$

Clearly $\theta = -1$. The compound character corresponding to

$$\{3^2\}+\{3\,2\,1\}+\{2^2\,1^2\}$$

of the symmetric group of order 6! for the respective classes

$$(1^6),\ (1^4\,2),\ (1^3\,3),\ (1^2\,4),\ (1^2\,2^2),\ (1\,2\,3),\ (1\,5),\ (6),\ (3^2),\ (2^3),\ (4\,2)$$

takes the values

$$30,\ -2,\ -3,\ 0,\ 2,\ 1,\ 0,\ 0,\ 0,\ -6,\ 0.$$

Hence the simple character $\chi^{(4\,3^2\,2)}$ for the classes

$$(2^6),\ (2^4\,4),\ (2^3\,6),\ (2^2\,8),\ (2^2\,4^2),\ (2\,4\,6),\ (2.10),\ (12),\ (6^2),\ (4^3),\ (8\,4)$$

takes the values

$$-30,\ 2,\ 3,\ 0,\ -2,\ -1,\ 0,\ 0,\ 0,\ 6,\ 0.$$

For the general case when the orders of the cycles in a class have a common factor $r$, the proofs and results are similar.

If $(\lambda)$ is a partition of $s$, and the numbers of terms of the sequence

$$\lambda_1+rm-1,\quad \lambda_2+rm-2,\quad \ldots,\quad \lambda_{rm}$$

congruent respectively to $0, 1, 2, \ldots, r-1$ to modulus $r$ are not all equal, then the $S$-function $\{\lambda\}$ of the roots of $f(x^r) = 0$, and the corresponding characteristics of all classes of the symmetric group of order $(rs)!$ in which the orders of all cycles are divisible by $r$, are zero.

Otherwise let the numbers of the sequence which are congruent to $q$ to modulus $r$ be

$$r[\mu_{q1}+m-1]+q,\quad r[\mu_{q2}+m-2]+q,\quad \ldots,\quad r\mu_{qm}+q.$$

Denote by $\psi$ the compound character of the symmetric group of order $S'$ corresponding to the product of the $S$-functions

$$\{\mu_{01}, \mu_{02}, \mu_{03}, \ldots, \mu_{0m}\},\quad \{\mu_{11}, \ldots, \mu_{1m}\},\quad \ldots,\quad \{\mu_{r-1,1}, \ldots, \mu_{r-1,m}\}.$$

Then if $\rho$ denotes the class $(1^{\alpha_1}2^{\alpha_2}\ldots)$ of the symmetric group of order $s!$, and $\rho'$ the class $(r^{\alpha_1}, (2r)^{\alpha_2}, (3r)^{\alpha_3}, \ldots)$ of the symmetric group of order $(rs)!$, we have

$$\chi_{\rho'}^{(\lambda)} = \theta\psi_{\rho},$$

where $\theta$ is $\pm 1$ according as the sequence

$$\begin{array}{llll} r[\mu_{r-1,1}+m-1]+r-1, & r[\mu_{r-2,1}+m-1]+r-2, & \ldots, & r[\mu_{01}+m-1], \\ r[\mu_{r-1,2}+m-2]+r-1, & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot \\ \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot\quad\cdot & \cdot\quad\cdot\quad\cdot & r\mu_{0m} \end{array}$$

is a positive or negative permutation of

$$\lambda_1+rm-1, \quad \lambda_2+rm-2, \quad \ldots, \quad \lambda_{rm}.$$

### Graphs and lattices

A method of using graphs and lattices for the evaluation of characters is given in Chapter V, § 5.3, especially Theorem II.

### Orthogonal properties

The most systematic check for the accuracy of a table of characters, whether of a symmetric or of any other finite group, is by the use of the orthogonal properties (4.2; 4, 4.2; 5).

The only errors that can evade a series of such checks, in general, are the complete interchange of two (or more) characters, or of two classes of equal order.

*Note to Page* 147.

Dr. J. A. Todd, F.R.S., informs me that he believes he has a gegenbeispiel which shows that the conjectured theorem at the foot of page 147 is not, in fact, true.

# IX

# GROUP CHARACTERS AND THE STRUCTURE OF GROUPS

**9.1.** In the following chapter (**1**) a method is given for the calculation of the table of characters of a subgroup when the characters of the group are known. Hence, since any group is a subgroup of some symmetric group, a comparatively simple method is obtained for the calculation of the characters of any group.

Further, it is shown how many of the important properties of a group may be deduced from its table of characters, and hence is revealed a novel method for the investigation of the structure of groups. At the end of the book, together with the tables of characters of the symmetric groups of orders less than or equal to 9!, is given a list of tables of characters of the more important subgroups.

Some of the processes are tentative in nature, but they have the merit of practical utility. The tables of characters of the subgroups, given at the end of this book, were calculated by the methods given here. Most methods known for the determination of subgroups are more or less tentative, and the method described here is no exception. However, many subgroups may be found by this method, almost by inspection, and the easiest subgroups to find are those with the largest orders, and these present most difficulty by other methods. The group of order 1,344, which is discussed later, is an instance.

There is a possibility of further theoretical developments that would render the methods much more definite, and make the exhaustive study of the subgroups of large orders quite possible.

For example, there is good reason to suppose, though rigorous proof is lacking, that the coefficients of the simple characters in the compound character corresponding to a maximal subgroup are all zero or unity. This would follow, if the following theorem could be proved:

*If $G$ is a subgroup of $H$, and $G$ has two linear invariants in an irreducible representation of $H$, then there is a subgroup $G_1$ of $H$, containing $G$, which has only one linear invariant.*

In this form the theorem is plausible, and it is further found to be true in all known cases. If it could be proved, the exhaustive study of the subgroups would be brought much nearer, for it is only necessary in the first instance to find the maximal subgroups, the other subgroups being found in turn from these.

Again, there is room for other discoveries concerning the possible compound characters, corresponding to subgroups, that might greatly simplify the work. For example, in the case of the symmetric group, if the compound character includes the character $[1^n]$, then the subgroup is contained in the alternating group, and the compound character is self-associated. The compound character must obey many similar laws, which, if known, would greatly simplify the work.

We have shown in § 3.8 that corresponding to any subgroup $G$ of order $g$ of a group $H$ of order $h$, there is a representation of $H$ as a permutation group of degree $\nu = h/g$, the group element $U$ corresponding to the permutation

$$UT_1 G, \quad UT_2 G, \quad \ldots, \quad UT_\nu G$$

of the sets

$$T_1 G, \quad T_2 G, \quad \ldots, \quad T_\nu G.$$

This permutation group is simply isomorphic with a set of permutation matrices, which gives a (reducible) representation of the group as a set of matrices. The spurs of the matrices, therefore, form a compound character of $H$.

Denote $T_i G$ by $A_i$.

Thus, corresponding to any subgroup there is a representation as a permutation group, a matrix representation, and a compound character $\phi$.

Also $\phi_0$, the spur of the unit matrix, is the number of rows or columns of the matrix, namely $\nu = h/g$.

We next find $\phi(T)$, the value of the compound character for the operation $T$.

Multiply

$$A_1, A_2, \ldots, A_\nu$$

on the left by $T$. We obtain

$$TA_1, TA_2, \ldots, TA_\nu.$$

Then $\phi(T)$ is the number of sets $A_r$ that are left unchanged. Let $A_r$ be any such set, and let $S_r$ be any operation of $A_r$.

Then

$$TA_r = A_r,$$
$$A_r = S_r A_1,$$
$$TS_r A_1 = S_r A_1,$$
$$S_r^{-1} T S_r A_1 = A_1.$$

Hence $S_r^{-1}TS_r$ belongs to $G$.

Corresponding to any set $A_r$ which is unchanged when multiplied on the left by $T$, there will be $g$ operations $S_r$ such that $S_r^{-1}TS_r$ belongs to $G$.

Conversely, if $S_r^{-1}TS_r$ belongs to $G$ and $S_r$ belongs to $A_r$,

$$TA_r = A_r.$$

In all there will be $g\phi(T)$ operations $S_r$ such that $S_r^{-1}TS_r$ belongs to $G$.

Let $T$ belong to a class $\rho$ containing $h_\rho$ operations of $H$, of which $g_\rho$ belong to $G$. If $S_r$ runs through the $h$ operations of $H$, $S_r^{-1}TS_r$ will run through the $h_\rho$ operations of $\rho$, taking the value of each operation of $\rho$ exactly $h/h_\rho$ times. Consequently, $hg_\rho/h_\rho$ of the operations $S_r^{-1}TS_r$ belong to $G$.

Thus

$$hg_\rho/h_\rho = g\phi(T),$$
$$\phi(T) = hg_\rho/gh_\rho.$$

It is clear, then, that from a knowledge of the subgroup and the number of operations in the classes, the compound character $\phi(T)$ may be obtained immediately.

Conversely, a knowledge of the character $\phi(T)$ defines the number of elements of each class that appear in the group $G$. This does not, of course, define the group $G$ uniquely, unless it is self-conjugate, for any transform of $G$ will correspond to the same character $\phi$. We shall show later how to deduce from the table of characters of $H$, and the compound character $\phi$, the table of characters of the subgroup $G$, and since it will further become apparent that most, if not all, of the properties of the group are deducible from the table of characters, it may be inferred that, corresponding to one compound character $\phi$, the corresponding subgroups, if not conjugate in $H$, are at least simply isomorphic.

The problem of finding the subgroups of a group, then, becomes the problem of finding suitable compound characters $\phi$. Before seeking methods for solving this problem, we pass on to the characters of the subgroups.

## 9.2. Deduction of the characters of a subgroup from those of the group

We make use of Frobenius's relations between the characters of a group and those of a subgroup. Let the group $H$ of order $h$ have a subgroup $G$ of order $g$, and let the class $\rho$ of $H$ contain $h_\rho$ operations, of which $g_\rho$ belong to $G$, and in $G$ these operations split up into classes $\rho'$, $\rho''$, etc., containing $g_{\rho'}$, $g_{\rho''}$,... operations respectively, so that

$$g_\rho = g_{\rho'}+g_{\rho''}+\dots.$$

Denote by $\chi_\rho^{(i)}$ a character of $H$ and by $\phi_{\rho'}^{(j)}$ a character of $G$.

Then Frobenius's relations are

$$\chi_\rho^{(i)} = \sum_i g_{ij}\phi_{\rho'}^{(j)}, \tag{9.2; 1}$$

$$\sum_{\rho'} \frac{hg_{\rho'}}{gh_\rho}\phi_{\rho'}^{(j)} = \sum_i g_{ij}\chi_\rho^{(i)}, \tag{9.2; 2}$$

the summation on the left of (9.2; 2) being taken over the classes $\rho'$, $\rho''$, etc., into which $\rho$ separates, and the coefficients $g_{ij}$, which are positive integers not greater than $\phi_0^{(j)}$ or $\chi_0^{(i)}$, being the same in both sets of equations.

Equation (9.2; 2) may be written

$$\frac{hg_\rho}{gh_\rho}\phi_\rho^{(j)} = \sum_i g_{ij}\chi_\rho^{(i)}, \tag{9.2; 3}$$

where $\phi_\rho^{(j)}$ is the mean character of the classes $\rho'$, $\rho''$, etc.

Now suppose that we know the table of characters of a group $H$, and the value of $hg_\rho/(gh_\rho)$ for each class of $H$.

Equation (9.2; 1) shows that the simple characters of $H$ are compound characters of $G$, and we obtain as many compound characters of $G$ as there are classes of $H$. If, however, a class of $H$ contains no operation of $G$, these compound characters will be linearly dependent. We obtain as many linearly independent compound characters of $G$ as there are classes of $H$ containing operations of $G$. If each class $\rho$ of $H$ corresponds to but one class of $G$, we have sufficient compound characters to determine all the characters of $G$.

In general, however, some of the classes $\rho$ of $H$ will separate into two or more classes $\rho'$, $\rho''$,... in $G$. For each extra class introduced thus, we lack one equation to determine the characters of $G$, for the characters of $H$ will only give us those compound characters of $G$ which take the same values for the classes $\rho'$, $\rho''$, etc. To provide the extra equations we use Frobenius's second equation.

First simplify the known compound characters as far as possible. The multiplicity of any compound character may be deduced from the orthogonal relations. The number of simple characters common to two characters may also be found. If one compound character entirely includes another, the latter may be subtracted, and a simpler character obtained. This will be illustrated in the example.

Now consider two compound characters of $G$, $\psi$ and $\psi'$, which have just one simple character in common. From (9.2; 3)

$$\frac{hg_\rho}{gh_\rho}\phi_\rho^{(j)} = \sum_i g_{ij}\chi_\rho^{(i)}, \tag{9.2; 4}$$

$\phi^{(j)}$ being the simple common character, in the case of classes of $H$ which separate into more than one class of $G$, $\phi_\rho^{(j)}$ representing the mean character. We need first to determine the coëfficients $g_{ij}$.

Since $\psi$ and $\psi'$ are compound characters,

$$\left.\begin{aligned} \frac{hg_\rho}{gh_\rho}\psi_\rho &= \sum p_i\chi_\rho^{(i)}, \\ \frac{hg_\rho}{gh_\rho}\psi'_\rho &= \sum q_i\chi_\rho^{(i)}. \end{aligned}\right\} \tag{9.2; 5}$$

$p_i$ and $q_i$ can be found as follows. Multiply the first equation of (9.2; 5) by $h_\rho\chi_\rho^{(k)}$ and sum for all the classes.

$$\sum_\rho h_\rho\frac{hg_\rho}{gh_\rho}\psi_\rho\chi_\rho^{(k)} = \sum p_i h_\rho\chi_\rho^{(i)}\chi_\rho^{(k)} = hp_k.$$

Hence
$$p_k = \frac{1}{g}\sum g_\rho\psi_\rho\chi_\rho^{(k)}, \tag{9.2; 6}$$

and, similarly,
$$q_k = \frac{1}{g}\sum g_\rho\psi'_\rho\chi_\rho^{(k)}.$$

Since $\psi$ and $\psi'$ both contain $\phi^{(j)}$,

$$p_i \geqslant g_{ij},$$
$$q_i \geqslant g_{ij}.$$

Hence $\sum g_{ij}\chi^{(i)}$ is contained in the common part of $\sum p_i\chi^{(i)}$ and $\sum q_i\chi^{(i)}$. In many cases $\sum g_{ij}\chi^{(i)}$ will be equal to the common part of $\sum p_i\chi^{(i)}$ and $\sum q_i\chi^{(i)}$, but it may happen that two simple characters of $G$, included respectively in $\psi$ and $\psi'$, correspond to two compound characters of $H$ that have a simple character in common. If this is so, this simple character also will be included in the common part of $\sum p_i\chi^{(i)}$ and $\sum q_i\chi^{(i)}$. However, even in this case the correct compound character $\sum g_{ij}\chi^{(i)}$ is not difficult to find, since, first, $\sum g_{ij}\chi_0^{(i)}$ must be divisible by $h/g$ from (9.2; 3), and, secondly, $\sum g_{ij}\chi^{(i)}$ must vanish for those classes of $H$ which contain no operations of $G$. These considerations usually determine $g_{ij}$ without ambiguity.

We obtain thus the simple character $\phi^{(j)}$, though only the mean value for those classes $\rho'$, $\rho''$, etc., which correspond to a single class of $H$. The equation
$$\sum g_{\rho'}\phi_{\rho'}^{(j)^2} = g$$
usually enables us to determine the actual values for these classes.

Example. The table of characters of the symmetric group of order 8! is given on p. 267. The compound character
$$\chi^{(8)}+\chi^{(4^2)}+\chi^{(2^4)}+\chi^{(1^8)}$$
corresponds to a permutation representation. Necessary criteria that a compound character should correspond to a permutation representation will be discussed later. For the present we assume it of this compound character. With this assumption, there must exist a subgroup of order
$$8!/\sum \chi_0 = 8!/30 = 1{,}344.$$

The compound character is zero for the negative classes, and for the positive classes takes the values

$$30,\quad 0,\quad 6,\quad 0,\quad 2,\quad 6,\quad 0,\quad 2,\quad 6,\quad 2,\quad 0,\quad 14,$$

respectively, for their order in the table referred to. Hence the numbers of operations of these classes in the subgroup are

$$1,\quad 0,\quad 42,\quad 0,\quad 168,\quad 224,\quad 0,\quad 384,\quad 252,\quad 224,\quad 0,\quad 49.$$

The simple characters of the symmetric group are compound characters of the subgroup, hence we obtain the following characters,

($\alpha$) $\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,\quad 1,$

for the eight existent classes of $G$. This is clearly simple.

($\beta$) $\quad 7,\quad 3,\quad 1,\quad 1,\quad 0,\quad -1,\ -1,\ -1.$

A rough approximation to $\sum g_\rho \psi_\rho^2$ for this character shows this to be about 1,300. Since it must be a multiple of the order of the group 1,344, it must be exactly 1,344, and the character is simple.

($\gamma$) $\quad 20,\quad 4,\quad 0,\quad -1,\ -1,\quad 0,\quad 1,\quad 4.$

Finding approximations to $\sum g_\rho \psi_\rho^{(\gamma)2}$ and $\sum g_\rho \psi_\rho^{(\beta)}\psi_\rho^{(\gamma)}$, we see that the character ($\gamma$) is the sum of two simple characters, neither of which is the character ($\beta$).

($\delta$) $\quad 21,\quad 1,\quad -1,\quad 0,\quad 0,\quad 1,\quad 0,\quad -3.$

This proves to be a simple character not contained in ($\gamma$).

($\epsilon$) 28, 4, 0, 1, 0, 0, $-1$, $-4$.

This is the sum of the two characters ($\beta$) and ($\delta$).

($\xi$) 64, 0, 0, $-2$, 1, 0, 0, 0.

This is the sum of four simple characters, and includes the character ($\delta$) once, since
$$\sum g_\rho \psi_\rho^{(\delta)} \psi_\rho^{(\xi)} = 1{,}344.$$

Hence, subtracting this character, we obtain

($\eta$) 43, $-1$, 1, $-2$, 1, $-1$, 0, 3,

which is the sum of three simple characters of which one is included in ($\gamma$), since
$$\sum g_\rho \psi_\rho^{(\gamma)} \psi_\rho^{(\eta)} = 1{,}344.$$

The next step is to separate the common character of ($\gamma$) and ($\eta$). From (9.2; 5),
$$\frac{hg_\rho}{gh_\rho} \psi_\rho^{(\gamma)} = \sum p_i \chi_\rho^{(i)},$$
and, from (9.2; 6),
$$p_i = \frac{1}{g} \sum g_\rho \psi_\rho^{(\gamma)} \chi_\rho^{(i)}.$$

$g_\rho \psi_\rho^{(\gamma)}$ takes the values

20, 168, 0, $-224$, $-384$, 0, 224, 196

for the eight classes, and multiplying these numbers by the values of each of the characters of $H$ in turn, and summing, an approximation again being sufficient, we see that $\sum p_i \chi^{(i)}$ includes the characters
$$2[6\,2]+[5\,2\,1]+[4^2]+[4\,3\,1]+2[4\,2^2]$$
together with the characters associated with each of these.

Similarly,
$$\frac{hg_\rho}{gh_\rho} \psi^{(\eta)} = \sum q_i \chi^{(i)}$$
includes the characters
$$[6\,2]+3[5\,2\,1]+[5\,1^3]+2[4\,3\,1]+3[4\,2^2],$$
together with the characters associated with these, and the self-associated character $2[4\,2\,1^2]$.

The common part of these is
$$[6\,2]+[5\,2\,1]+[4\,3\,1]+2[4\,2^2]+\text{associated characters.}$$
The corresponding $\chi_0$ is
$$(20+64+70+2\times 56)\times 2.$$

This is not divisible by $h/g = 30$. By inspection it is apparent that the character $[4\,2^2]$ cannot be repeated in $\sum g_{ij}\chi^{(i)}$ if $\sum g_{ij}\chi_0^{(i)}$ is to be divisible by 30.

The compound character

$$[6\,2]+[5\,2\,1]+[4\,3\,1]+[4\,2^2]+\text{associated characters}$$

must be the one required, and, as a check, this vanishes for the classes of $H$ containing no operations of $G$. We are thus led to the simple character of $G$ with mean values, for the classes of $H$,

$$(\zeta) \qquad 14,\ \ 2,\ \ 0,\ \ -1,\ \ 0,\ \ -2/3,\ \ 1,\ \ 10/7.$$

It is clear that the sixth and last classes must split into separate classes. $h_\rho/g_{\rho'}$ must divide $h/g$. Hence the sixth class must divide into two classes of orders 84 and 168, with characters 0 and $-1$, or else $-2$ and 0. Other possibilities are excluded by the equation $\sum g_\rho \psi_\rho^{(\zeta)2} = 1{,}344$, the values 2 and $-2$, for example, making the left-hand side exceed the right.

Similarly, the last class must split into two. The orders of these classes may be 42 and 7, with characters 2, $-2$, or 1, 4; or again 35 and 14, with characters 2, 0, or 0, 5; or lastly 28 and 21 with characters 1, 2, or 4, $-2$.

The only solution which satisfies

$$\sum g_\rho \psi_\rho^{(\zeta)2} = 1{,}344$$

is that the sixth class should split into two classes of orders

$$84 \quad \text{and} \quad 168$$

with characters $\qquad -2$ and $0$,

and the last class into two classes of orders

$$42 \quad \text{and} \quad 7$$

with characters $\qquad 2$ and $-2$.

We obtain thus five simple characters of $G$. Since we have introduced two extra classes we need one more character not linearly dependent on the characters of $H$. We obtain this by squaring the character $(\zeta)$. The square of a character is always a character, simple or compound. This, together with the remaining characters of $H$, yields, without difficulty, the following characters of $G$:

| Order of Class | 1 | 42 | 168 | 224 | 384 | 168 | 84 | 224 | 42 | 7 |
|---|---|---|---|---|---|---|---|---|---|---|
| Character (*a*) . . | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| (*b*) . . | 6 | 2 | 0 | 0 | −1 | 0 | 2 | 0 | 2 | 6 |
| (*c*) . . | 7 | 3 | 1 | 1 | 0 | −1 | −1 | −1 | −1 | −1 |
| (*d*) . . | 14 | 2 | 0 | −1 | 0 | 0 | −2 | 1 | 2 | −2 |
| (*e*) . . | 21 | 1 | −1 | 0 | 0 | 1 | 1 | 0 | −3 | −3 |
| (*f*) . . | 7 | −1 | −1 | 1 | 0 | 1 | −1 | −1 | 3 | −1 |
| (*g*) . . | 21 | −3 | 1 | 0 | 0 | −1 | 1 | 0 | 1 | −3 |
| (*h*) . . | 7 | −1 | −1 | 1 | 0 | −1 | −1 | 1 | −1 | 7 |
| (*j*) . . | 8 | 0 | 0 | −1 | 1 | 0 | 0 | −1 | 0 | 8 |
| (*k*)+(*l*) . . | 6 | −2 | 2 | 0 | −1 | 2 | −2 | 0 | −2 | 6 |

All these characters are simple save the last, which is the sum of two simple characters.

For each class, if the characters are real,

$$\sum \chi_\rho^{(\lambda)2} = g/g_\rho.$$

For complex characters this equation is replaced by

$$\sum \chi_\rho^{(\lambda)}\overline{\chi_\rho^{(\lambda)}} = g/g_\rho.$$

It is clear that the class of order 384 must split into two classes to provide the extra class, since 384 does not divide 1,344. We obtain then, for the characters (*k*) and (*l*),

$$(k) \qquad 3,\ -1,\ 1,\ 0,\ \frac{-1-\sqrt{7}i}{2},\ \frac{-1+\sqrt{7}i}{2},\ 1,\ 1,\ 0,\ -1,\ 3.$$

$$(l) \qquad 3,\ -1,\ 1,\ 0,\ \frac{-1+\sqrt{7}i}{2},\ \frac{-1-\sqrt{7}i}{2},\ 1,\ 1,\ 0,\ -1,\ 3.$$

The characters take the complex values for the two classes of order 192, into which the class of order 384 divides. This is the only way of satisfying the above orthogonal relation.

Alternatively, the characters (*k*) and (*l*) could be obtained separately by using the characters of the alternating group instead of those of the symmetric group.

We thus complete the table of characters (see p. 276).

## 9.3. Determination of subgroups: Necessary criteria that a compound character should correspond to a permutation representation of the group

From the table of characters of the group we may obtain any compound character. If we can ascertain that any given compound character corresponds to a permutation representation of the group, we can immediately deduce the existence of a conjugate set of subgroups, and the number of operations from each class of the group that belong to one of the subgroups. Further, as shown in the last section, we can deduce the table of characters of the subgroup.

All that we require, then, to enable us to investigate the structure of a group from its table of characters is a set of criteria that will enable us to decide whether any given compound character corresponds to a permutation representation. We set forth the following criteria in order of simplicity, since the use of the criteria in this order reduces the labour of investigation to a minimum. Let $\phi$ be a compound character. The following criteria must be satisfied if it corresponds to a permutation representation.

CRITERION I. $\phi_p$ *is a positive or zero integer for every class.*

CRITERION II. $\phi(S^r) \geqslant \phi(S)$ *for all indices r.*

Those symbols left unchanged by a permutation corresponding to the operation $S$ will be left unchanged by any power of $S$.

It follows from this, for example, that the value of the character for the class (6) of the symmetric group of order 6! cannot exceed the value for either of the classes $(3^2)$ or $(2^3)$. Again, for the symmetric group of order 7!, the character $[7]+[4\,2\,1]$ cannot correspond to a permutation representation, for its value for the class (6 1) is 2, and for the class $(3^2\,1)$ is zero. To the table of characters may be added extra columns for the differences $\chi(S^2)-\chi(S)$, $\chi(S^3)-\chi(S)$, etc., for those classes for which these may take negative values. In seeking a compound character that is positive or zero for every class, these columns may be taken into account.

CRITERION III. $\phi_0$ *divides the order of the group.*

CRITERION IV. *If* $\phi = \sum g_i \chi^{(i)}$, *then* $g_i \leqslant \chi_0^{(i)}$.

The group matrix corresponds to the group of permutations of the $h$ operations of the group. It corresponds to the smallest possible subgroup, namely, the identical element itself and the compound character

$$\phi = \sum \chi_0^{(i)} \chi^{(i)}.$$

This permutation group is, in general, imprimitive, the elements being divided into sets corresponding to any subgroup of $H$. It follows that the matrices corresponding to a permutation group corresponding to any such subgroup may be obtained by a reduction of the group matrix. Hence

$$g_i \leqslant \chi_0^{(i)}.$$

Criterion V. *$\phi$ contains the character that is unity for every operation exactly once.*

Since $\phi$ is essentially positive,

$$\sum g_\rho \phi_\rho > 0.$$

Hence $\phi$ must contain this character at least once. It cannot contain it more than once by Criterion IV.

Criterion VI. *$\phi_\rho h_\rho/\phi_0$ is integral for every class, and if the group is symmetric, and the class $\rho$ contains a cycle of prime order $p$, the integer is divisible by $p-1$.*

$\phi_\rho h_\rho/\phi_0$ is the number of operations of the class in the subgroup, and hence must be integral. Further, if the group is symmetric, and the class contains a cycle of prime order $p$, the number of operations from this class in the subgroup must be divisible by $p-1$, for the powers of any operation $S$ of the subgroup must also belong to the subgroup. The latter part does not hold if the group is not symmetric, since the powers of $S$ containing the same cycles as $S$ may belong to a different class.

Criterion VII. *The table of characters of the subgroup corresponding to the compound character $\phi$ may be found after the method of the previous section.*

This last criterion is the only one that may be taken as final and sufficient. The existence of a consistent table of characters may be taken as sufficient evidence for the existence of the subgroup. The calculation of the characters of the subgroup is a comparatively long process, and should not be attempted unless the compound character is found to satisfy the other six criteria.

One other criterion may be added as a final check.

Criterion VIII. *The product of two characters or the square of any character is a linear function of the characters with positive integral coefficients.*

Example. We proceed to find those subgroups of the symmetric group of order 8! of which the orders are greater than 1,000.

We have to find suitable compound characters $\phi$ such that

$$\phi_0 < 8!/1{,}000 < 41.$$

Referring to the table of characters of the symmetric group of order 8!, which is given on p. 267, we see that the inequality $\phi_0 < 41$

together with Criteria I and II give a series of diophantine inequalities with a very small number of solutions. A short examination only is sufficient to show that the only solutions which satisfy Criteria III, V, and VI are

$$\begin{aligned}\phi^{(1)} &= \chi^{(8)}+\chi^{(1^8)},\\ \phi^{(2)} &= \chi^{(8)}+\chi^{(7\,1)},\\ \phi^{(3)} &= \chi^{(8)}+\chi^{(7\,1)}+\chi^{(6\,2)},\\ \phi^{(4)} &= \chi^{(8)}+\chi^{(6\,2)}+\chi^{(4^2)},\\ \phi^{(5)} &= \chi^{(8)}+\chi^{(4^2)}+\chi^{(2^4)}+\chi^{(1^8)},\\ \phi^{(6)} &= \chi^{(8)}+\chi^{(7\,1)}+\chi^{(2\,1^6)}+\chi^{(1^8)}.\end{aligned}$$

$\phi^{(1)}$ corresponds to the alternating group.

Remembering that

$$\{7\}\{1\} = \{8\}+\{7\,1\},$$

we see that $\phi^{(2)}$ corresponds to an intransitive subgroup, the symmetric group on 7 symbols.

Again $$\{6\}\{2\} = \{8\}+\{7\,1\}+\{6\,2\}.$$

Hence $\phi^{(3)}$ corresponds to an intransitive subgroup which is the direct product of the symmetric groups on 6 and 2 symbols respectively.

$\phi^{(6)}$ corresponds to the alternating group on 7 symbols, for the group is clearly the positive subgroup of the group corresponding to $\phi^{(2)}$.

$\phi^{(5)}$ would correspond to a subgroup of order 1,344. It clearly satisfies the first six criteria. Since we have, in the last section, obtained the characters of the corresponding subgroup we may take this as sufficient evidence for the existence of the subgroup.

This group is one of the simplest to discover by the method which is given here. By the usual methods of group theory, however, it is by no means so easily found. It was omitted in the first lists of the subgroups of the symmetric groups, as published by Cayley† in 1891, and by Cole‡ in 1893. Its existence was, however, known to Jordan.||

Lastly, $\phi^{(4)}$ corresponds to a subgroup of order $8!/35 = 1{,}152$. The table of characters may be found as shown in the last section (see p. 277), and, as before, the successful completion of the table of characters may be taken as sufficient evidence for the existence of the group.

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† Cayley (23). ‡ Cole (25). || Jordan (42).

Referring again to the group of order 1,344, we see by inspection that

$$\phi = \chi^{(a)}+\chi^{(b)},$$
$$\phi = \chi^{(a)}+\chi^{(c)},$$
$$\phi = \chi^{(a)}+\chi^{(f)},$$
$$\phi = \chi^{(a)}+\chi^{(h)}$$

satisfy the first six criteria.

There is no difficulty in showing that corresponding to these there is a subgroup of order 1,344/7 = 192, and three distinct sets of subgroups of order 1,344/8 = 168, two of which are simply isomorphic, for we obtain the same character table, though the classes in the two cases are derived from different classes of the original group, the third being a distinct group.

*By this method, not only may the subgroups be found, but, as we shall see, many of the properties of the subgroups may be read directly from their tables of characters.*

## 9.4. The properties of groups and character tables

### Self-conjugate subgroups

A necessary and sufficient condition that a group should possess a self-conjugate subgroup is that it should possess a character $\chi$, not unity for every class, and a class $\rho$, other than the class containing identity, such that

$$\chi_\rho = \chi_0.$$

It is further necessary that characters $\chi^{(1)}, \chi^{(2)},..., \chi^{(p)}$ exist such that, if $\phi$ represents the compound character

$$\phi = \chi_0^{(1)}\chi^{(1)}+\chi_0^{(2)}\chi^{(2)}+...+\chi_0^{(p)}\chi^{(p)},$$

then
$$\phi_\rho = \phi_0,$$
or
$$\phi_\rho = 0,$$

for every class. The character $\phi$ corresponds to the self-conjugate subgroup which consists of the operations of those classes for which

$$\phi_\rho = \phi_0.$$

Suppose that the group $H$ has a self-conjugate subgroup $G$, corresponding to the compound character $\phi$. From the class $\rho$ of $H$ there are $\phi_\rho h_\rho/\phi_0$ operations in $G$. If this is neither zero nor $h_\rho$, then an operation of the class $\rho$ belonging to $G$ may be transformed into an operation not belonging to $G$, by an operation of $H$, and $G$ cannot be self-conjugate.

Hence $$\phi_\rho h_\rho/\phi_0 = h_\rho \text{ or } 0,$$

i.e. $$\phi_\rho = \phi_0,$$

or $$\phi_\rho = 0.$$

If $$\phi = \sum g_i \chi^{(i)},$$

then $$\chi_\rho^{(i)} = \chi_0^{(i)}$$

for all classes $\rho$ of $H$ whose operations belong to $G$.

Hence the first condition is necessary. We now show that it is sufficient.

Let $\chi$ be a character of $H$, not equal to unity for every class, and let $\rho$ be a class, not containing identity, such that

$$\chi_\rho = \chi_0.$$

Let $S$ be an operation of the class $\rho$, and let $P$ be the corresponding matrix in a matrix representation of the group corresponding to the character $\chi$. Then $P$ is a matrix of degree $\chi_0$ and with spur $\chi_0$.

If the order of the operation $S$ is $p$, then

$$S^p = I,$$

$$P^p = I.$$

Hence the $\chi_0$ roots of the characteristic equation of $P$ are all $p$th roots of unity, and since the sum of the $\chi_0$ roots is the spur of $P = \chi_0$, the $\chi_0$ roots must all be unity, and

$$P = I.$$

The set of operations $S$ of $H$, which correspond to the unit matrix $I$ in this representation, must form a self-conjugate subgroup $G$ of $H$. The first condition is sufficient.

Lastly, suppose that the self-conjugate subgroup $G$ of order $g$, of the group $H$ of order $h$, exists, and corresponds to the compound character

$$\phi = \sum g_i \chi^{(i)}.$$

Denote by $\mathscr{G}$ the sum of the operations of $G$. Then

$$\mathscr{G}^2 = g\mathscr{G}.$$

But $$\sum \phi(S).S = h\mathscr{G}/g,$$

the summation being taken over the group $H$, and

$$\sum \phi(S).S = \sum g_i \chi^{(i)}(S).S.$$

Now $(\chi_0^{(i)}/h) \sum \chi^{(i)}(S).S$ is the modulus of an invariant sub-algebra of the group algebra, and is consequently idempotent. Further, the product of two such expressions corresponding to different characters is zero.

Hence
$$\{\sum \chi^{(i)}(S).S\}^2 = \frac{h}{\chi_0^{(i)}} \sum \chi^{(i)}(S).S$$

and
$$\{\sum \phi(S).S\}^2 = \sum \frac{hg_i^2}{\chi_0^{(i)}} \chi^{(i)}(S).S.$$

But
$$\begin{aligned}\{\sum \phi(S).S\}^2 &= \{h\mathscr{G}/g\}^2 \\ &= h^2\mathscr{G}/g \\ &= h \sum \phi(S).S.\end{aligned}$$

Hence
$$\sum \frac{hg_i^2}{\chi_0^{(i)}} \chi^{(i)}(S).S = \sum hg_i \chi^{(i)}(S).S,$$

and
$$g_i = \chi_0^{(i)}.$$

Thus
$$\phi = \chi_0^{(i)}\chi^{(i)},$$

completing the proof that the second condition is necessary.

EXAMPLES. The symmetric group of order 4! has the following table of characters:[†]

| Class | $1^4$ | $1^2 2$ | $1 3$ | $4$ | $2^2$ |
|---|---|---|---|---|---|
| Order | 1 | 6 | 8 | 6 | 3 |
| $[4]$ | 1 | 1 | 1 | 1 | 1 |
| $[3\,1]$ | 3 | 1 | 0 | $-1$ | $-1$ |
| $[2^2]$ | 2 | 0 | $-1$ | 0 | 2 |
| $[2\,1^2]$ | 3 | $-1$ | 0 | 1 | $-1$ |
| $[1^4]$ | 1 | $-1$ | 1 | $-1$ | 1 |

The self-conjugate subgroups correspond to
$$\phi = \chi^{(4)} + \chi^{(1^4)},$$
which is the alternating group comprising the first, third, and fifth classes, and to
$$\phi = \chi^{(4)} + 2\chi^{(2^2)} + \chi^{(1^4)}.$$
This group is of order $24/(1+4+1) = 4$, and includes the identical operation and the three operations of the fifth class.

Consider once again the group of order 1,344, the table of characters of which is given on p. 155. The characters $\chi^{(a)}$, $\chi^{(b)}$, $\chi^{(h)}$, $\chi^{(j)}$, $\chi^{(k)}$, and $\chi^{(l)}$ satisfy
$$\chi_p = \chi_0$$
for the fourth class of order 7.

Corresponding to
$$\phi = \chi^{(a)} + 6\chi^{(b)} + 7\chi^{(h)} + 8\chi^{(j)} + 3\chi^{(k)} + 3\chi^{(l)},$$

---

[†] For cycles, classes, and all partitions $1\,3$, $1^2\,2$ denote the partitions 1, 3, and 1, 1, 2 of 4 respectively, etc.

we have a self-conjugate subgroup of order

$$1{,}344/(1+36+49+64+9+9) = 8,$$

which comprises the identical operation and the seven operations of the fourth class.

## The direct product of two groups

Let $G_1$ and $G_2$ be two groups of orders $g_1$ and $g_2$ respectively, and let $H$ be the direct product of the two groups.

Suppose that $G_1$ has $p$ classes $\rho_1$ and that $G_2$ has $q$ classes $\rho_2$. Then, if $T$ belongs to $H$ and

$$T = S_1 S_2,$$

where $S_1$ belongs to $G_1$ and $S_2$ to $G_2$, the class of $T$ depends on the classes of $S_1$ and $S_2$. Hence there will be $pq$ classes of $H$. If $S_1$ belongs to the class $\rho_1$ and $S_2$ to $\rho_2$, $T$ will belong to a class $\rho$, and we may write symbolically

$$\rho = \rho_1 \rho_2.$$

If $h_\rho$, $g_{\rho_1}$, and $g_{\rho_2}$ are the orders of these classes, then

$$h_\rho = g_{\rho_1} g_{\rho_2}.$$

Now the direct product of two matrix representations of $G_1$ and $G_2$ respectively must give a matrix representation of $H$. The spur of the direct product of two matrices is the product of the spurs.

Hence, if $\chi_{\rho_1}^{(i)}$ is a character of $G_1$, and $\chi_{\rho_2}^{(j)}$ is a character of $G_2$,

$$\chi_{\rho_1 \rho_2}^{(ij)} = \chi_{\rho_1}^{(i)} \chi_{\rho_2}^{(j)} \tag{9.4;1}$$

is a character of $H$.

Further, $$\sum h_\rho \chi_\rho^{(ij)2} = \sum g_{\rho_1} \chi_{\rho_1}^{(i)2} \sum g_{\rho_2} \chi_{\rho_2}^{(j)2}$$
$$= g_1 g_2 = h.$$

Hence the character is simple, and we obtain thus the $pq$ characters of $H$.

The equation (9.4; 1), then, gives the complete table of characters of the direct product of two groups.

It is possible to infer from the table of characters of a group, almost by inspection, whether it can be expressed as the direct product of two groups.

To begin with, there must be two perfectly distinct self-conjugate subgroups, which may be distinguished as shown in the last paragraph. If this condition is satisfied, we must see if the whole table may be expressed in the form

$$\chi_{\rho_1 \rho_2}^{(ij)} = \chi_{\rho_1}^{(i)} \chi_{\rho_2}^{(j)},$$

the classes $\rho_1$ belonging to the first subgroup, and the classes $\rho_2$ to the second; the characters $\chi^{(i)}$ being those that satisfy $\chi_\rho^{(i)} = \chi_0^{(i)}$ for the classes of the second subgroup, and the characters $\chi^{(j)}$ those which satisfy $\chi_\rho^{(j)} = \chi_0^{(j)}$ for the classes of the first subgroup.

If this condition is satisfied, then the group is the direct product of two groups with character tables $[\chi_\rho^{(i)}]$ and $[\chi_\rho^{(j)}]$ respectively.

EXAMPLE. From the table of characters of the symmetric group of order 7!, we obtain a subgroup of order 144, corresponding to the compound character

$$\chi^{(7)}+\chi^{(61)}+\chi^{(52)}+\chi^{(43)}.$$

The table of characters of this group, which may be found as shown in § 4, is as follows:

| Class | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ | $I$ | $J$ | $K$ | $L$ | $M$ | $N$ | $P$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 6 | 3 | 8 | 2 | 6 | 3 | 18 | 24 | 12 | 16 | 9 | 18 | 6 | 12 |
| $a$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $b$ | 2 | 2 | 0 | 2 | −1 | 2 | 2 | 0 | 0 | −1 | −1 | 0 | 0 | −1 | −1 |
| $c$ | 1 | 1 | −1 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 |
| $d$ | 3 | 1 | 3 | 0 | 3 | −1 | −1 | 1 | 0 | 1 | 0 | −1 | −1 | −1 | −1 |
| $e$ | 6 | 2 | 0 | 0 | −3 | −2 | −2 | 0 | 0 | −1 | 0 | 0 | 0 | 1 | 1 |
| $f$ | 3 | 1 | −3 | 0 | 3 | −1 | −1 | −1 | 0 | 1 | 0 | 1 | 1 | −1 | −1 |
| $g$ | 2 | 0 | 2 | −1 | 2 | 0 | 2 | 0 | −1 | 0 | −1 | 2 | 0 | 2 | 0 |
| $h$ | 4 | 0 | 0 | −2 | −2 | 0 | 4 | 0 | 0 | 0 | 1 | 0 | 0 | −2 | 0 |
| $i$ | 2 | 0 | −2 | −1 | 2 | 0 | 2 | 0 | 1 | 0 | −1 | −2 | 0 | 2 | 0 |
| $j$ | 3 | −1 | 3 | 0 | 3 | 1 | −1 | −1 | 0 | −1 | 0 | −1 | 1 | −1 | −1 |
| $k$ | 6 | −2 | 0 | 0 | −3 | 2 | −2 | 0 | 0 | 1 | 0 | 0 | 0 | 1 | −1 |
| $l$ | 3 | −1 | −3 | 0 | 3 | 1 | −1 | 1 | 0 | −1 | 0 | 1 | −1 | −1 | 1 |
| $m$ | 1 | −1 | 1 | 1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 |
| $n$ | 2 | −2 | 0 | 2 | −1 | −2 | 2 | 0 | 0 | 1 | −1 | 0 | 0 | 1 | −1 |
| $p$ | 1 | −1 | −1 | 1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | −1 |

There is a self-conjugate subgroup which comprises the classes $A$, $B$, $D$, $F$, and $G$, and another comprising the classes $A$, $C$, and $E$. The corresponding characters are $a$, $d$, $g$, $j$, and $m$, and $a$, $b$, and $c$.

Hence, if the group is the direct product of these two subgroups, then the characters of these subgroups must be:

| Order of Class | 1 | 6 | 8 | 6 | 3 |
|---|---|---|---|---|---|
| | 1 | 1 | 1 | 1 | 1 |
| | 3 | 1 | 0 | −1 | −1 |
| | 2 | 0 | −1 | 0 | 2 |
| | 3 | −1 | 0 | 1 | −1 |
| | 1 | −1 | 1 | −1 | 1 |

| Order of Class | 1 | 3 | 2 |
|---|---|---|---|
| | 1 | 1 | 1 |
| | 2 | 0 | −1 |
| | 1 | −1 | 1 |

It is easily verified that the table of characters of the group may be obtained from these two tables of characters of the subgroup by means of the equation

$$\chi^{(ij)}_{\rho_1\rho_2} = \chi^{(i)}_{\rho_1}\chi^{(j)}_{\rho_2}.$$

Also the orders of the classes of the group are the products of the orders of classes of the two subgroups.

The tables of characters of the subgroups are easily recognized as those of the symmetric groups of orders 4! and 3! respectively. Hence the group is the direct product of two symmetric groups of orders 4! and 3! respectively.

This might, of course, have been deduced at the beginning from the compound character

$$\phi = \chi^{(7)}+\chi^{(61)}+\chi^{(52)}+\chi^{(43)}.$$

Since $$\{4\}\{3\} = \{7\}+\{61\}+\{52\}+\{43\},$$

a subgroup of the symmetric group of order 7!, corresponding to the compound character $\phi$, must be the direct product of two symmetric groups of orders 4! and 3! respectively. The object of the example, however, was to illustrate the deduction of the property from the character table.

## 9.5. Transitivity

### The group reduction function

Let $G$ be a subgroup of order $g$ and compound character $\phi$ of the symmetric group $H$ of order $h = n!$.

We associate with $G$ a symmetric function of weight $n$ which we call the *group reduction function* (2), which we shall write shortly G.R.F.

Definition. *If the compound character associated with $G$ is $\phi = \sum k^{(\lambda)}\chi^{(\lambda)}$, then the G.R.F. of $G$ is $\sum k^{(\lambda)}\{\lambda\}$.*

The G.R.F. of $G$ indicates immediately the number of elements of $G$ in any class of $H$, for if $\rho$ denotes the class $(1^\alpha 2^\beta 3^\gamma ...)$ and $g_\rho$ elements of this class belong to $G$, then, if the G.R.F. of $G$ is expressed in terms of the $S_r$'s, we have

$$\text{G.R.F. of } G = \frac{1}{g}\sum g_\rho S_\rho,$$

where $S_\rho = S_1^\alpha S_2^\beta S_3^\gamma ...$.

This result follows immediately from the properties of $S$-functions.

It is easily seen that the G.R.F. of the direct product of two groups is the product of their G.R.F.s.

Two groups with the same G.R.F. are, in general, transforms of one another.

The G.R.F. of the symmetric group of order $n!$ is $h_n$; of the corresponding alternating group, $h_n+a_n$.

It is sometimes convenient to refer to a group with a given G.R.F. instead of a group with a given compound character. In many cases these amount to different terminologies conveying the same essential facts.

### Transitivity

The forms of the simple characters which occur in the compound character of a subgroup, or, equivalently, the coefficients of the various $S$-functions in the G.R.F., are intimately connected with the transitivity of the group, as will be seen from the following section. To examine the transitivity of a given subgroup possessing a known compound character, we compare it with certain known subgroups and find the *transitive factor* (3) of the two groups.

Let $G_1$ and $G_2$ be two subgroups of orders $g_1$ and $g_2$ respectively of the symmetric group $H$ of order $h = n!$. Let $\Gamma$ denote the direct product of the groups $G_1$, $G_2$, and let $\gamma \equiv (S, T)$ be an element of $\Gamma$ corresponding to the elements $S$ and $T$ of $G_1$ and $G_2$ respectively. Let $Z_r$ $(r = 1, 2,..., h)$ denote the $h$ operations of $H$. Then, if $Z_r$ is replaced by $S^{-1}Z_r T$, we obtain, corresponding to $\gamma$, a permutation of the $h$ operations $Z_r$. We obtain thus a representation of $\Gamma$, which we shall denote by $\bar{\Gamma}$, as a permutation group on the $h$ symbols $Z_r$.

The representation $\bar{\Gamma}$ of $\Gamma$ is not, in general, transitive.

DEFINITION. *The number of transitive sets into which the symbols* $Z_r$ *are divided in the permutation representation* $\bar{\Gamma}$ *of* $\Gamma$ *is called the* transitive factor *of the two groups, and is denoted by* $N(G_1, G_2)$.

We now proceed to evaluate $N(G_1, G_2)$. Let $\gamma \equiv (S, T)$ be an element of $\Gamma$ which leaves the symbol $Z_r$ invariant, so that

$$S^{-1}Z_r T = Z_r,$$

$$Z_r^{-1}SZ_r = T.$$

Clearly $S$ and $T$ belong to the same class $\rho$ of $H$.

Suppose there are $p$ elements of $\Gamma$ which leave $Z_r$ invariant in this manner. Let $\gamma_1 \equiv (S_1, T_1)$ be any other element of $\Gamma$ which replaces $Z_r$ by $Z_s$, so that

$$S_1^{-1} Z_r T_1 = Z_s.$$

Then clearly

$$(SS_1)^{-1} Z_r (TT_1) = Z_s,$$

and $\gamma\gamma_1$ also replaces $Z_r$ by $Z_s$.

Conversely, if $\gamma_2 \equiv (S_2, T_2)$ replaces $Z_r$ by $Z_s$, then

$$S_2 Z_s T_2^{-1} = Z_r,$$

$$(S_1 S_2^{-1})^{-1} Z_r T_1 T_2^{-1} = Z_r,$$

and $\gamma_1\gamma_2^{-1}$ is one of the $p$ elements which leave $Z_r$ invariant.

Hence exactly $p$ elements of $\Gamma$ replace $Z_r$ by $Z_s$, and by the $g_1 g_2$ elements of $\Gamma$, $Z_r$ is replaced by $g_1 g_2/p$ different elements of $H$.

Denote by $\theta(Z_r)$ the number of elements of $\Gamma$ which leave $Z_r$ invariant, so that

$$\theta(Z_r) = p.$$

Then $\sum \theta(Z_r)$ summed for the $g_1 g_2/p$ elements of the transitive set to which $Z_r$ belongs is clearly equal to $g_1 g_2$. Hence

$$\sum_1^h \theta(Z_r) = g_1 g_2 \text{ times number of transitive sets}$$

$$= g_1 g_2 N(G_1, G_2).$$

Now if $\gamma \equiv (S, T)$ leaves $Z_r$ invariant,

$$S^{-1} Z_r T = Z_r,$$

$$Z_r^{-1} S Z_r = T,$$

and $S$ and $T$ belong to the same class $\rho$ of $H$. Let the order of the class be $h_\rho$, and let the number of elements from it in $G_1$ and $G_2$ respectively be $g_{\rho_1}$ and $g_{\rho_2}$.

The $h$ elements $Z_r$ transform $S$ into the $h_\rho$ elements of $\rho$, each being repeated $h/h_\rho$ times, hence there are $h/h_\rho$ elements $Z_r$ which transform $S$ into $T$. For a given class $\rho$ there are $g_{\rho_1} g_{\rho_2}$ pairs of elements $S$, $T$ which can be transformed into one another.

Clearly, then,

$$\sum \theta(Z_r) = \sum_\rho g_{\rho_1} g_{\rho_2} h/h_\rho,$$

summed for the classes $\rho$. Hence

$$N(G_1, G_2) = \frac{1}{g_1 g_2} \sum \theta(Z_r)$$

$$= \sum_\rho \frac{g_{\rho_1} g_{\rho_2} h}{g_1 g_2 h_\rho}.$$

Now let the compound characters of $H$ corresponding to the subgroups $G_1$ and $G_2$ be $\phi_1 = \sum k_1^{(\lambda)}\chi^{(\lambda)}$ and $\phi_2 = \sum k_2^{(\lambda)}\chi^{(\lambda)}$ respectively. Then

$$\phi_{1\rho} = \frac{hg_{\rho_1}}{g_1 h_\rho},$$

$$\phi_{2\rho} = \frac{hg_{\rho_2}}{g_2 h_\rho}.$$

Hence
$$N(G_1, G_2) = \sum \frac{h_\rho}{h}\phi_{1\rho}\phi_{2\rho}.$$

But since
$$\sum h_\rho \chi_\rho^{(\lambda)2} = h,$$
$$\sum h_\rho \chi_\rho^{(\lambda)}\chi_\rho^{(\mu)} = 0 \quad ((\lambda) \neq (\mu)),$$

we have
$$N(G_1, G_2) = \frac{1}{h}\sum h_\rho k_1^{(\lambda)}k_2^{(\mu)}\chi_\rho^{(\lambda)}\chi_\rho^{(\mu)}$$
$$= \sum_\lambda k_1^{(\lambda)}k_2^{(\lambda)}.$$

I. *If the compound characters of the subgroups $G_1$ and $G_2$ of $H$ are $\phi_1 = \sum k_1^{(\lambda)}\chi^{(\lambda)}$ and $\phi_2 = \sum k_2^{(\lambda)}\chi^{(\lambda)}$ respectively, then*
$$N(G_1, G_2) = \sum_\lambda k_1^{(\lambda)}k_2^{(\lambda)}.$$

The theorem might be alternatively expressed:

*If $G_1$ and $G_2$ are two permutation groups on the same symbols with G.R.F.s $\sum k_1^{(\lambda)}\{\lambda\}$ and $\sum k_2^{(\lambda)}\{\lambda\}$ respectively, then*
$$N(G_1, G_2) = \sum k_1^{(\lambda)}k_2^{(\lambda)}.$$

This result may be used to study the relations between the G.R.F. of a group (or its compound character) and its transitive properties.

Take for $G_1$ any assigned group $G$ with given G.R.F., $\sum k^{(\lambda)}\{\lambda\}$. For $G_2$ we shall take in turn certain simple intransitive groups.

Let $H$ permute the symbols $\alpha, \beta, \gamma, \delta, \epsilon, \ldots, \zeta$. First take $G_2$ to be the group with G.R.F. $h_{n-1}h_1$ which leaves $\alpha$ invariant, but permutes the other symbols symmetrically.

If $G$ is a transitive subgroup of $H$, an operation $S$ of $G$ may be found which replaces $\alpha$ by any given symbol. Since an operation $T$ of $G_2$ may be found which permutes the other $n-1$ symbols in any manner, it is clear that by a suitable choice of $S$ and $T$, $S^{-1}Z_r T$ will represent any given operation of $H$. In other words, the permutation group $\bar{\Gamma}$ is transitive, and

$$N(G, G_2) = 1.$$

Since the G.R.F. of $G_2$ is

$$h_{n-1}h_1 = \{n\}+\{n-1, 1\},$$

the following theorem holds:

*The necessary and sufficient condition that a group $G$ is transitive is that its G.R.F. does not include the S-function $\{n-1, 1\}$.*

If $G$ is intransitive, by counting the number of transitive sets in $G$, and therefore in $\bar{\Gamma}$, we obtain

*If $G$ is intransitive, the number of transitive sets into which the symbols are divided is one more than the coefficient of $\{n-1, 1\}$ in the G.R.F. of $G$.*

For doubly transitive groups we take $G_2$ as a group with G.R.F. $h_{n-2}h_1^2$. Suppose $G_2$ leaves the symbols $\alpha$ and $\beta$ invariant and permutes the rest symmetrically. Then if $G$ is doubly transitive an operation $S$ of $G$ can be found which replaces $\alpha$ and $\beta$ by any assigned symbols, and since an operation $T$ of $G_2$ can be found which permutes the other symbols in any manner, it follows as before that $\bar{\Gamma}$ is transitive.

*The necessary and sufficient condition that $G$ is doubly transitive is that its G.R.F. does not include the S-functions $\{n-1, 1\}$, $\{n-2, 2\}$, $\{n-2, 1^2\}$.*

Generally, by taking $G_2$ to be a group with G.R.F. $h_{n-r}h_1^r$ we have

*If $\lambda_1$ is the greatest part in any partition of an S-function included in the G.R.F. of $G$, excluding the term $\{n\}$, then $G$ is $(n-\lambda_1-1)$-ply transitive.*

We now give an example which further illustrates the relations between the transitive properties of a group, and the $S$-functions included in its G.R.F.

The octahedral group, i.e. the group of rotations which rotate a regular octahedron into itself, is well known to be a group of order 24 simply isomorphic with the symmetric group of that order.† It may be regarded as a permutation group $G$ on the six vertices of the octahedron, and as such is a subgroup of the symmetric group of order 6!. We will deduce by the above method the G.R.F. of this permutation group.

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† Denote the four pairs of opposite faces by $A, A'$; $B, B'$; $C, C'$, and $D, D'$. It is easily seen that the pairs of faces are permuted symmetrically, which gives the required representation.

The G.R.F. includes the term $\{6\}$ as does that of every permutation group of degree 6.

The group is transitive, and $\{5\,1\}$ is absent.

Denote the three pairs of diametrically opposite vertices by $A$, $A'$; $B$, $B'$; $C$, $C'$. Then the number of transitive sets of pairs of vertices is two, since these two vertices may be adjacent, as e.g. $A$ and $B$, or opposite, as e.g. $A$ and $A'$. Furthermore, this number of transitive sets is unaltered if permutations of the pair of vertices is allowed, since operations of the group $G$ itself interchange them.

Hence taking $G_2$ as the subgroup with G.R.F.

$$h_4 h_1^2 = \{6\}+2\{5\,1\}+\{4\,2\}+\{4\,1^2\},$$

and $G_3$ as the subgroup with G.R.F.

$$h_4 h_2 = \{6\}+\{5\,1\}+\{4\,2\},$$

we have
$$N(G, G_2) = N(G, G_3) = 2.$$

Clearly the G.R.F. of $G$ includes the term $\{4\,2\}$ but not $\{4\,1^2\}$.

There are five transitive sets of three vertices which may be typified by

$$ABC,\quad ACB,\quad BAA',\quad ABA',\quad AA'B.$$

If the symmetric group is allowed to permute these three vertices, the number of transitive sets is reduced to two. If the alternating group only is allowed to permute them, the number is three, as the group $G$ itself will interchange $A$ and $A'$, and the last three go into the same transitive set.

Take $G_4$, $G_5$, and $G_6$ as three groups with G.R.F.

$$h_3 h_1^3 = \{6\}+3\{5\,1\}+3\{4\,2\}+3\{4\,1^2\}+\{3^2\}+2\{3\,2\,1\}+\{3\,1^3\},$$
$$h_3^2 = \{6\}+\{5\,1\}+\{4\,2\}+\{3^2\},$$
$$h_3(h_3+a_3) = \{6\}+\{5\,1\}+\{4\,2\}+\{3^2\}+\{4\,1^2\}+\{3\,1^3\},$$

respectively.

Then
$$N(G, G_4) = 5,$$
$$N(G, G_5) = 2,$$
$$N(G, G_6) = 3.$$

Clearly, in the G.R.F. of $G$ the term $\{3^2\}$ is absent, $\{3\,1^3\}$ has coefficient 1, and $\{3\,2\,1\}$ is absent.

Lastly, consider the number of ways of dividing the six vertices into three sets of two, irrespective of order in each set. There are six transitive sets of such divisions, which may be typified by

$$\begin{array}{lll} AA', & BB', & CC'; \\ AA', & BC, & B'C'; \\ AB, & A'B', & CC'; \\ AB, & A'C, & B'C'; \\ AB, & A'C', & B'C; \\ AB, & CC', & A'B'; \end{array}$$

Taking $G_7$ as a subgroup with G.R.F.

$$h_2^3 = \{6\}+2\{5\,1\}+3\{4\,2\}+\{3^2\}+2\{3\,2\,1\}+\{2^3\},$$

we have

$$N(G, G_7) = 6.$$

Clearly, the G.R.F. of $G$ includes the term $\{2^3\}$ with coefficient 2.

The coefficient of $S_1^6$ in the expression

$$\{6\}+\{4\,2\}+\{3\,1^3\}+2\{2^3\}$$

is

$$(1+9+10+2\,.\,5)/6! = 1/24 = 1/g.$$

Hence there is no other term, and this is the G.R.F. of $G$.

The G.R.F. of $G$ could also have been obtained by counting the operations with given cycles, but the object of the example was to illustrate the method.

### Characters of intransitive subgroups

The intransitive subgroups of the symmetric group of degree $n = p+q+r+\dots$, that leave sets of $p$, $q$, $r$,... symbols invariant, are subgroups of the direct product of the symmetric groups of substitutions on $p$, $q$, $r$,... symbols respectively. Hence, to find these subgroups and their characters, we may start with the table of characters of this group, instead of the symmetric group on $n$ symbols.

More generally, for particular intransitive subgroups, any of these symmetric groups in the direct product may be replaced by one of its subgroups.

The table of characters is given by

$$\chi^{(ijk\dots)}_{pp'p''\dots} = \chi^{(i)}_{p}\chi^{(j)}_{p'}\chi^{(k)}_{p''}\dots,$$

$\chi^{(i)}$ being a character of the symmetric group on $p$ symbols, or a subgroup of this, $\chi^{(j)}$ on $q$ symbols, $\chi^{(k)}$ on $r$ symbols, etc.

From the table of characters we may seek subgroups corresponding to compound characters

$$\phi = \sum g_{ijk\dots}\,\chi^{(ijk\dots)}.$$

If such a compound character can be expressed in the form

$$\phi = \sum g_i \chi^{(i)} \sum g_{jk\ldots} \chi^{(jk\ldots)},$$

then the subgroup is the direct product of a group of substitutions on $p$ symbols and an intransitive subgroup on $n-p$ symbols leaving sets of $q, r,\ldots$ symbols invariant.

It is sometimes more convenient to start with the direct product of two such groups and seek compound characters which cannot be expressed in the form

$$\phi = \sum g_i \chi^{(i)} \sum g_j \chi^{(j)}.$$

Example. We seek subgroups of the symmetric group of order 6!, which leave two sets of three symbols invariant.

The table of characters of the direct product of two symmetric groups, each of order 3!, is as follows:

| Class | $\rho_1\sigma_1$ | $\rho_1\sigma_2$ | $\rho_1\sigma_3$ | $\rho_2\sigma_1$ | $\rho_2\sigma_2$ | $\rho_2\sigma_3$ | $\rho_3\sigma_1$ | $\rho_3\sigma_2$ | $\rho_3\sigma_3$ |
|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 3 | 2 | 3 | 9 | 6 | 2 | 6 | 4 |
| (*a*) [3.3] | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| (*b*) [3.21] | 2 | 0 | −1 | 2 | 0 | −1 | 2 | 0 | −1 |
| (*c*) [$3.1^3$] | 1 | −1 | 1 | 1 | −1 | 1 | 1 | −1 | 1 |
| (*d*) [21.3] | 2 | 2 | 2 | 0 | 0 | 0 | −1 | −1 | −1 |
| (*e*) [21.21] | 4 | 0 | −2 | 0 | 0 | 0 | −2 | 0 | 1 |
| (*f*) [$21.1^3$] | 2 | −2 | 2 | 0 | 0 | 0 | −1 | 1 | −1 |
| (*g*) [$1^3.3$] | 1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 |
| (*h*) [$1^3.21$] | 2 | 0 | −1 | −2 | 0 | 1 | 2 | 0 | −1 |
| (*j*) [$1^3.1^3$] | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

There are three subgroups of order 18, corresponding to the characters $(a)+(c)$, $(a)+(g)$, $(a)+(j)$, of which the first two may be expressed as direct products of two groups of degree 3.

There are two subgroups of order 12, corresponding to $(a)+(b)$ and $(a)+(d)$, which may be expressed as direct products.

There are three subgroups of order 6, corresponding to $(a)+(e)+(j)$, $(a)+(b)+(h)+(j)$, and $(a)+(d)+(f)+(j)$. The last two groups are really of degree 5, each being the positive subgroup of the direct product of symmetric groups of orders 3! and 2!, the other symbol being left invariant.

All other subgroups are of degree less than 6.

## 9.6. Invariant subgroups

### Characters of quotient groups

Suppose that a group $H$ of order $h$ has an invariant subgroup $G$ of order $g$. We proceed to show how to deduce the characters of the quotient group $\Gamma = H/G$ from the character table of $H$.

If we also use $G$ and $H$ to represent the sums of the operations of the respective groups, we may find $\nu = h/g$ operations of $H$, $S_1 = I$, $S_2,\ldots, S_\nu$, such that

$$S_1 G+S_2 G+\ldots+S_\nu G = H,$$

where $S_r$ may be replaced by any operation of the set $S_r G$ without altering the set, and

$$S_r G = GS_r.$$

Now

$$G^2 = gG,$$

and

$$S_i GS_j G = S_i S_j G^2$$
$$= gS_k G,$$

where $S_i S_j$ belongs to the set $S_k G$.

The quantities

$$\frac{1}{g}S_1 G, \quad \frac{1}{g}S_2 G, \quad \ldots, \quad \frac{1}{g}S_\nu G$$

form a group which is the quotient group $\Gamma$.

In any matrix representation of $\Gamma$, the quantity $(1/g)S_r G$ is represented by $M_r$. We obtain thus a matrix representation of the whole group $H$ by representing every operation of the set $S_r G$ by $M_r$. Every operation of $G$ will be represented by the unit matrix.

Hence every character $\chi^{(\lambda)}$ of $\Gamma$ is a character of $H$ in which all classes $\rho$ containing operations of $G$ satisfy

$$\chi_\rho^{(\lambda)} = \chi_0^{(\lambda)}.$$

Again, $\sum \chi_0^{(\lambda)}\chi^{(\lambda)}$, summed for all the characters of $H$ whose values are the same for all operations of $G$, is the compound character of $H$ corresponding to the subgroup $G$.

Hence $\sum \chi_0^{(\lambda)2} = h/g = \nu$, and these characters must give the complete table of characters of the quotient group $\Gamma$.

Classes of $H$ which take the same values for all these characters $\chi^{(\lambda)}$ correspond to the same class of $\Gamma$. The number of operations in a class of $\Gamma$ is equal to the total number of operations in the classes of $H$ that go to form this class, divided by $g$.

EXAMPLE. The table of characters of the symmetric group of order 4! is given on p. 265.

Referring to this, we observe that there is an invariant subgroup of order 4 corresponding to $[4]+2[2^2]+[1^4]$, and another of order 12 corresponding to $[4]+[1^4]$.

The orders of the classes of the first quotient group are

$$(1+3)/4 = 1, \quad (6+6)/4 = 3, \quad 8/4 = 2.$$

We obtain the table of characters

| Order of class | 1 | 3 | 2 |
|---|---|---|---|
| [4] | 1 | 1 | 1 |
| $[2^2]$ | 2 | 0 | −1 |
| $[1^4]$ | 1 | −1 | 1 |

The orders of the classes of the second quotient group are

$$(1+8+3)/12 = 1, \qquad (6+6)/12 = 1.$$

We obtain the character table

| Order of class | 1 | 1 |
|---|---|---|
| [4] | 1 | 1 |
| $[1^4]$ | 1 | −1 |

Hence the two quotient groups are the symmetric groups of orders 3! and 2! respectively.

**Commutator subgroups and perfect groups**

If $S$ and $T$ are operations of a group $H$, the operation $S^{-1}T^{-1}ST$ is called a commutator. The commutators of a group generate an invariant subgroup called the commutator subgroup.

The quotient group of a group $H$ corresponding to its commutator subgroup is Abelian. Conversely, every invariant subgroup which is complementary to an Abelian quotient group includes the commutator subgroup.†

An Abelian group of order $n$ has $n$ characters for each of which $\chi_0 = 1$. Conversely, if a group of order $n$ has $n$ characters for which $\chi_0 = 1$, the group is Abelian. For an Abelian group must have as many classes as operations, and hence as many characters, and, conversely, such a group must be Abelian.

Hence, following the last paragraph, it follows that from the table of characters of a group we may immediately deduce the commutator subgroup. For the corresponding compound character is

$$\phi = \sum \chi^{(\lambda)},$$

summed for all characters $\chi^{(\lambda)}$ satisfying

$$\chi_0^{(\lambda)} = 1.$$

---

† Miller, Blichfeldt, and Dickson (7).

The quotient group complementary to this subgroup must be Abelian, for all its characters satisfy $\chi_0 = 1$. Hence the group contains the commutator subgroup. Also, no smaller invariant subgroup can be found of which the quotient group is Abelian. Hence this must be the commutator subgroup.

A group is perfect if it is identical with its commutator subgroup. Hence the condition that a group is perfect is that it possesses no character satisfying $\chi_0 = 1$ save that character which is unity for every operation.

## Solvable groups

A group is said to be solvable if and only if it contains a series of invariant subgroups, such that the last of the series is identity, and the index of each of these invariant subgroups under the next larger invariant subgroup is a prime number.† Every Abelian group is solvable.

The solvability of a group may be deduced from its table of characters as follows.

Let $H$ be a group of which the characters are known, and let $G_1$ be its commutator subgroup. Deduce the table of characters of $G_1$ as has been shown.

From the table of characters of $G_1$ deduce the commutator subgroup $G_2$ and its table of characters.

Proceeding thus, we arrive either at a perfect group $G_r$ or at an Abelian group $G_r$ with commutator subgroup the identity. In the first case the group $H$ is insolvable, in the second case solvable.

The quotient group at each stage is Abelian, and hence solvable. A perfect group is insolvable, and any group containing an insolvable subgroup is insolvable.

In this process, if the simple characters of $H$ lead to compound characters of a subgroup, it is not necessary to separate these compound characters, as at each stage the commutator subgroup is invariant in $H$, and takes complete classes and complete characters from $H$. It must be remembered, however, that the character is compound, in deducing the next commutator subgroup.

Example. The symmetric group of order 6! has a subgroup of order 72 corresponding to the compound character $\phi = \chi^{(6)}+\chi^{(42)}$. Its table of characters is as follows:

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† Miller, Blichfeldt, and Dickson (7), 174.

| Class | $A$ | $B$ | $C$ | $D$ | $E$ | $F$ | $G$ | $H$ | $J$ |
|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 6 | 4 | 9 | 12 | 12 | 4 | 6 | 18 |
| $(a)$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $(b)$ | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 |
| $(c)$ | 1 | 1 | 1 | 1 | 1 | $-1$ | 1 | $-1$ | $-1$ |
| $(d)$ | 1 | $-1$ | 1 | 1 | $-1$ | 1 | 1 | 1 | $-1$ |
| $(e)$ | 4 | 2 | 1 | 0 | $-1$ | 0 | $-2$ | 0 | 0 |
| $(f)$ | 4 | $-2$ | 1 | 0 | 1 | 0 | $-2$ | 0 | 0 |
| $(g)$ | 4 | 0 | $-2$ | 0 | 0 | $-1$ | 1 | 2 | 0 |
| $(h)$ | 4 | 0 | $-2$ | 0 | 0 | 1 | 1 | $-2$ | 0 |
| $(j)$ | 2 | 0 | 2 | $-2$ | 0 | 0 | 2 | 0 | 0 |

The commutator subgroup corresponds to the compound character $(a)+(b)+(c)+(d)$, and consists of the classes $A$, $C$, $D$, and $G$. It has the following characters. $(a)$ is simple. $(j)$ is a simple character taken twice over. $(e)$ and $(g)$ are each the sum of two characters.

The next commutator subgroup corresponds to the compound character $(a)+\frac{1}{2}(j)$, and consists of the classes $A$, $C$, and $G$. At this stage we obtain the characters

| Class | $A$ | $C$ | $G$ |
|---|---|---|---|
| Order | 1 | 4 | 4 |
| $(a)$ | 1 | 1 | 1 |
| $(e)$ | 4 | 1 | $-2$ |
| $(g)$ | 4 | $-2$ | 1 |

$(e)$ and $(g)$ are each the sum of four simple characters. Hence the group is Abelian, and the original group of order 72 is solvable.

**Contragredient isomorphisms.** (See p. 177)

No case is known of a contragredient isomorphism which changes each class into itself. On the other hand, it is still an open question whether or no such isomorphisms exist.†

If a group contains no invariant subgroup, its group of cogredient isomorphisms is simply isomorphic with the group itself. The existence of contragredient isomorphisms that interchange some of the classes may be deduced from the table of characters. Hence, apart from the questionable existence of contragredient isomorphisms that leave all classes invariant, a complete knowledge of the isomorphisms may be obtained from the character table.

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† Burnside (1).

A contragredient isomorphism, not leaving all classes invariant, must correspond to an interchange of the classes. By reason of the isomorphism, however, the table of characters must be left unchanged, and, since all the characters cannot take the same values for two different classes, there must also be an interchange of characters. Further, the classes that are interchanged must have the same number of operations, and these operations must have the same order.

Hence, if a group admits a contragredient isomorphism, not leaving all classes invariant, then there is a possible rearrangement of the classes, which may be expressed as a series of cyclic interchanges of classes containing the same number of operations of the same order, and a rearrangement of the characters, which, taken together, leaves the table of characters unchanged.

Conversely, if such a rearrangement is possible, there exists a contragredient isomorphism.

The quotient group of the group of isomorphisms by the group of cogredient isomorphisms (or the set which leaves the classes invariant, if this is different) is the group of rearrangements of the classes under the above conditions, for if two contragredient isomorphisms $T$ and $U$ give rise to the same interchange of classes, then $T^{-1}U$ leaves all the classes unchanged.

Examples. In the table of characters of the symmetric group of order 6! (see p. 266), interchange the classes $(1^3\,3)$ and $(3^2)$, $(1^4\,2)$ and $(2^3)$, $(1\,2\,3)$ and $(6)$, and simultaneously the characters $[5\,1]$ and $[2^3]$, $[4\,1^2]$ and $[3\,1^3]$, $[3^2]$ and $[2\,1^4]$. The table of characters remains unchanged, and there is a contragredient isomorphism. This is the only rearrangement satisfying the above conditions, and the quotient group of the isomorphisms by the cogredient set is a group of order 2.

For the group of order 1,344 with table of characters on p. 276, a rearrangement of the classes and characters is possible satisfying all the conditions save that the orders of the operations are different. This condition failing, no contragredient isomorphism exists.

Consider the group of order 18,

$$S^9 = I, \quad T^2 = I, \quad ST = TS^8.$$

If $\alpha = 2\cos 40°$, $\beta = 2\cos 80°$, $\gamma = 2\cos 160°$, the table of characters is as follows:

| Class | $a$ | $b$ | $c$ | $d$ | $e$ | $f$ |
|---|---|---|---|---|---|---|
| Order | 1 | 2 | 2 | 2 | 2 | 9 |
| $A$ | 1 | 1 | 1 | 1 | 1 | 1 |
| $B$ | 1 | 1 | 1 | 1 | 1 | $-1$ |
| $C$ | 2 | $\alpha$ | $\beta$ | $\gamma$ | $-1$ | 0 |
| $D$ | 2 | $\gamma$ | $\alpha$ | $\beta$ | $-1$ | 0 |
| $E$ | 2 | $\beta$ | $\gamma$ | $\alpha$ | $-1$ | 0 |
| $F$ | 2 | $-1$ | $-1$ | $-1$ | 2 | 0 |

The table is left unchanged if we interchange cyclically the classes $(b)$, $(c)$, and $(d)$, and simultaneously the characters $[C]$, $[D]$, and $[E]$. Hence the quotient group of the group of isomorphisms by the cogredient set is a cyclic group of order 3.

*Note to Page* 175, *Contragredient isomorphisms*

Contragredient isomorphisms do exist which leave all the classes invariant. See Wall, *Supplementary Bibliography* (26).

# X

# CONTINUOUS MATRIX GROUPS AND INVARIANT MATRICES

**10.1.** THE operations of the tetrahedral, octahedral, and icosahedral groups rotate the respective regular solids into themselves. The operations of any of these groups are three-dimensional rotations. The complete set of rotations in three dimensions also forms a group which will rotate a sphere into itself. This group is of infinite order, and has the property of continuity associated with the real numbers. The group of all non-singular linear transformations in three variables, or more generally in $n$ variables, as well as many of its subgroups, is of a similar nature. Such groups may be represented as matrix groups. The elements of the matrix members are not restricted to a finite set of discrete values as for finite groups, but may be varied continuously over given ranges, subject, usually, to certain restrictive conditions. These will be referred to as *continuous matrix groups.*

Continuous matrix groups have many of the properties of finite groups. We shall now extend the group-character theory to these. The number of classes of such a group is infinite, and we may expect that the number of characters or of independent matrix representations is infinite also. But whereas the classes have in general a continuous nature, the matrix representations form in general a discrete enumerable set.

It will be found that the orthogonal properties hold for certain restricted groups, the summation over the elements of the group being replaced by an integration over the group manifold.

We deal first with the group of all non-singular matrices of order $n^2$.

### Invariant matrices (1)

Let $A = [a_{st}]$ be a non-singular matrix of order $n^2$. Let $G$ denote the group of all such matrices. $A$ is the matrix of a linear transformation

$$X = AY,$$

where $X$ and $Y$ are column vectors with elements $x_1, x_2,..., x_n$ and $y_1, y_2,..., y_n$ respectively.

Under the transformation the homogeneous products of the $x_r$'s of degree $p$, e.g.

$$x_1^p,\ x_2^p,\ \ldots,\ x_n^p,\ x_1^{p-1}x_2,\ \ldots,\ x_1x_2x_3\ldots x_p,\ \ldots,$$

are transformed into linear functions of similar products in the $y$'s by a matrix of order $\binom{n+p-1}{p}^2$.

This matrix is called the $p$th *induced matrix* of $A$, and will be denoted by $A^{[p]}$.

Quite clearly, if $$AB = C,$$
then $$A^{[p]}B^{[p]} = C^{[p]}. \qquad (10.1;1)$$

The elements of the matrix $A^{[p]}$ are homogeneous polynomials of degree $p$ in the elements of $A$.

Again, if $p$ ($< n$) sets of $n$ variables are transformed simultaneously by the matrix $A$,
$$X^{(r)} = AY^{(r)} \quad (1 \leqslant r \leqslant p),$$
we can form $\binom{n}{p}$ determinantal forms
$$|x_{e_t}^{(s)}| \quad (1 \leqslant e_1 < e_2 < \ldots < e_p \leqslant n).$$

Under the transformation these will be transformed into linear functions of a similar set of determinantal forms in the $y$'s.

The matrix of the transformation on these determinantal forms is a matrix of order $\binom{n}{p}^2$ in which the elements are the determinants of the $p$-rowed minors of $A$. It is called the $p$th *compound matrix* of $A$, and will be denoted by the symbol $A^{(p)}$.

Clearly, if $$AB = C,$$
then $$A^{(p)}B^{(p)} = C^{(p)}. \qquad (10.1;2)$$

By way of illustration it will be seen that the second induced matrix of
$$\begin{bmatrix} a, b \\ c, d \end{bmatrix}$$
is
$$\begin{bmatrix} a^2 & ab & b^2 \\ 2ac & ad+bc & 2bd \\ c^2 & cd & d^2 \end{bmatrix}$$
whilst the second compound matrix has the single element
$$ad-bc.$$

The induced and compound matrices are special cases of *invariant matrices* which have the following definition.

DEFINITION. *Let $T(A)$ be a matrix whose elements are polynomials in the elements of the matrix $A$. Then if*

$$T(A)T(B) = T(AB), \qquad (10.1;3)$$

*$T(A)$ is said to be an invariant matrix of the matrix $A$.*

The induced and compound matrices are clearly invariant matrices. There are, however, other invariant matrices. For example, the direct product of two invariant matrices is clearly an invariant matrix; hence the direct product of two induced matrices, of two compound matrices, or of an induced and a compound matrix, is an invariant matrix.

Let $T(A)$ be any invariant matrix. Then if $A$ runs through the operations of the matrix group $G$, the matrices $T(A)$ form a matrix representation of the group. In conformity with the theory for finite groups we define the characters of the group $G$ as follows.

If $A$ is a variable element of the group $G$ and $T(A)$ is an invariant matrix of $A$, the set of spurs of the matrices $T(A)$ is said to form the character of the group $G$ corresponding to the matrix representation $T(A)$.

If $T(A)$ is an invariant matrix of $A$, and $B$ is any constant non-singular matrix of the same order as $T$, then $B^{-1}T(A)B$ is also an invariant matrix. The two invariant matrices $T(A)$ and $B^{-1}T(A)B$ are said to be *equivalent*.

If $A$ is a non-singular matrix, but $T(A)$ is singular, then

$$T(A)T(A^{-1}) = T(I),$$

where $I$ is the unit matrix, is singular also. But $T(I)$ is idempotent and may be transformed into the form $\operatorname{diag}(1^p, 0^q)$. The corresponding transform of $T(A)$ has a non-singular matrix of order $p^2$ in the top left-hand corner, and zeros elsewhere, since

$$T(A) = T(I)T(A) = T(A)T(I).$$

We may ignore the zeros and consider only the non-singular matrix. Henceforward we shall assume all invariant matrices to be non-singular unless otherwise stated.

An invariant matrix $T(A)$ is said to be *reducible* when it can be transformed into the form

$$B^{-1}T(A)B = \left[\begin{array}{c|c} T_1(A) & M(A) \\ \hline 0 & T_2(A) \end{array}\right],$$

where $T_1(A)$ and $T_2(A)$ are square matrices. Clearly $T_1(A)$ and $T_2(A)$ are also invariant matrices. If $M(A) = 0$ also, $T(A)$ is said to be *completely reducible*, and it is thereby transformed into the direct sum of the two invariant matrices $T_1(A)$, $T_2(A)$.

$$\begin{aligned} B^{-1}T(A)B &= \left[\begin{array}{c:c} T_1(A) & 0 \\ \hdashline 0 & T_2(A) \end{array}\right] \\ &= T_1(A) \dotplus T_2(A), \end{aligned}$$

where $\dotplus$ is used to denote the direct sum indicated.

It will be shown in the next chapter that if an invariant matrix is reducible it is also completely reducible, and in this chapter by *reducible* we shall mean *completely reducible.*

If $T(A)$ is not reducible it is said to be irreducible. As for matrix representations of a finite group, we study the irreducible invariant matrices. The character of an irreducible invariant matrix is said to be *simple.*

I. *The elements of an irreducible invariant matrix $T(A)$ are homogeneous in the elements of $A$.*

Let $I$ be the unit matrix, and $\lambda$ a scalar. Then

$$T^2(\lambda I) = T(\lambda^2 I).$$

It follows that the characteristic roots of $T(\lambda I)$ are powers of $\lambda$. Let $B$ be a matrix which transforms $T(\lambda I)$ into diagonal form, so that

$$T'(A) = B^{-1}T(A)B,$$

and

$$T'(\lambda I) = \operatorname{diag}[(\lambda^{n_1})^{p_1}, (\lambda^{n_2})^{p_2},\dots, (\lambda^{n_i})^{p_i}].$$

Since

$$T'(A)T'(\lambda I) = T'(\lambda I)T'(A),$$

it follows that $T'(A)$ is the direct sum of matrices of orders $p_1^2$, $p_2^2,\dots, p_i^2$, these being each homogeneous and of degrees $n_1, n_2,\dots, n_i$ respectively in the elements of $A$. The theorem follows.

II. *The necessary and sufficient condition that two irreducible invariant matrices $T(A)$, $T'(A)$ are equivalent is that they should have the same spur for every matrix $A$.*

The condition is clearly necessary, as the spur of a transform of $T$ is equal to the spur of $T$. Let $A_1, A_2,\dots, A_r$ be an indefinite number of matrices of order $n^2$. Suppose that the spurs of $T(A_i)$, $T'(A_i)$ are equal for all matrices $A_i$. Then the spurs of $\sum x_i\, T(A_i)$, $\sum x_i\, T'(A_i)$

are equal. Again the spur of $[\sum x_i T(A_i)]^2$, which is equal to the spur of

$$\sum x_i^2 T(A_i^2) + \sum x_i x_j T(A_i A_j),$$

will be equal to the spur of $[\sum x_i T'(A_i)]^2$. Similarly, the spurs of all the powers of these matrices will be equal. It follows that the characteristic equations of $\sum x_i T(A_i)$, $\sum x_i T'(A_i)$ are equal.

We may deduce that the matrices $\sum x_i T(A_i)$, $\sum x_i T'(A_i)$ may be transformed into one another, provided that their characteristic equations have no repeated factors, and the corresponding representations are equivalent.

If, however, the characteristic equation of $\sum x_i T(A_i)$ has a repeated factor, or if it factorizes at all, for a sufficiently large value of $r$, then $T(A)$ will be reducible, contrary to hypothesis. This proves the sufficiency of the condition.

The theorem that reducibility implies complete reducibility, which will be proved in the next chapter, enables us to omit the word 'irreducible' from the enunciation of this theorem.

Invariant matrices may be constructed by the following method which Schur further elaborated to obtain the general theory (2). We shall not, however, follow Schur's development, but mention it only as an example.

We use $a_r$, $h_r$ and $\{\lambda\}$ to denote the symmetric functions and $S$-functions of the characteristic roots of the original matrix $A$. The spur of $A$ is thus $h_1$. The spurs of $A^{[r]}$ and $A^{(r)}$ are easily seen to be $h_r$ and $a_r$ respectively.

The direct product of $A$ with itself, $A \times A$, is an invariant matrix which is found to be reducible, and equivalent to the direct sum of the second induced and the second compound matrices,

$$A \times A = A^{[2]} \dotplus A^{(2)}.$$

The corresponding equation for the characters of these invariant matrices is

$$a_1^2 = h_2 + a_2.$$

We shall find it convenient to denote the $n$th compound matrix by $A^{[1^n]}$ instead of $A^{(n)}$.

The direct product of the $r$th induced matrix and the original matrix, $A^{[r]} \times A$, is also reducible. If the variables $x_i$ and $x_i'$ are transformed cogrediently by the matrix $A$, then $A^{[r]} \times A$ is the matrix of transformation of expressions in the form of a product of the

$x_i$'s of degree $r$ multiplied by some $x'_j$. If we put $x_i = x'_i$ for all $i$, certain linear functions of these expressions are zero, and the remaining expressions are transformed by the $(r+1)$th induced matrix. Hence $A^{[r]}\times A$ is equivalent to the direct sum of $A^{[r+1]}$ and some other invariant matrix which is the matrix of transformation on those linearly independent forms which become zero when $x_i = x'_i$. We denote this invariant matrix, which is found to be irreducible, by $A^{[r,1]}$. We have
$$A^{[r]}\times A = A^{[r+1]}+A^{[r,1]}.$$

The corresponding equation of the characters is
$$h_r h_1 = h_{r+1}+\{r, 1\}.$$
The character of the invariant matrix $A^{[r,1]}$ is thus $\{r, 1\}$.

Following this method, Schur has shown that there are invariant matrices corresponding to each $S$-function. These are irreducible and constitute a complete set of independent irreducible invariant matrices.

In the place of Schur's development we adopt the much simpler and more direct method of constructing a specific invariant matrix corresponding to each partition of $n$, showing this to be non-singular and irreducible, and to have its character equal to the corresponding $S$-function (3).

III. *If $A$ is a matrix of order $m^2$ there are as many irreducible invariant matrices of $A$, of degree $n$, as there are partitions of $n$ into not more than $m$ parts, and the spurs of these invariant matrices are the $S$-functions of weight $n$ of the characteristic roots of $A$.*

The theorem is proved by the construction of an invariant matrix corresponding to each partition, which is shown to be non-singular and irreducible and to have its spur equal to the corresponding $S$-function of the characteristic roots of $A$.

That no other independent irreducible invariant matrices of degree $n$ exist, follows from the fact that the spur of an invariant matrix of degree $n$ is a symmetric function of the characteristic roots of $A$, and can always be expressed as a sum of $S$-functions. Hence, since two invariant matrices are equivalent if they have the same spur, the invariant matrix is equivalent to the direct sum of the corresponding invariant matrices.

Let $(\sigma) = (1^{\sigma_1}, 2^{\sigma_2},..., m^{\sigma_m})$, $(\sigma_1+\sigma_2+...+\sigma_m = n)$, denote any $n$ of the numbers 1, 2,..., $m$. $(\tau) = (1^{\tau_1}, 2^{\tau_2},..., m^{\tau_m})$ is any other such set, or the same set. $A_{\sigma\tau}$ is the $n$-rowed minor of the matrix $A = [a_{st}]$

in which the rows are defined by the set of numbers $(\sigma)$, and the columns by the set $(\tau)$.

Then, if $A_{\sigma\tau} = [q_{st}]$ and $S$ is the permutation $e_1, e_2,\ldots, e_n$ of the numbers $1, 2,\ldots, n$, we define $P_{\sigma\tau}(S)$ by the equation

$$P_{\sigma\tau}(S) = q_{1e_1} q_{2e_2} \cdots q_{ne_n}/r_\sigma,$$

where $r_\sigma = \sigma_1!\,\sigma_2!\ldots\sigma_m!$.

We make use of the matrix representation of the symmetric group given by Theorem VI, § 5.4. On substituting for $P_i$, $M_i$, $\sigma_{ij}$, etc., suppose that this theorem gives

$$e_{ij} = \frac{f^{(\lambda)}}{n!} \sum_S \phi_{ij}^{(\lambda)}(S).S,$$

where $S$ is an arbitrary element of the symmetric group and $\phi_{ij}^{(\lambda)}(S)$ is a scalar coefficient.

From Theorem VII, § 5.4, we obtain

$$S = \sum \phi_{ji}^{(\lambda)}(S^{-1})e_{ij}. \tag{10.1;4}$$

IV. *The matrix*

$$T(A) = [\textstyle\sum \phi_{ts}^{(\lambda)}(S^{-1}).P_{\sigma\tau}(S)],$$

*where $s$ and $(\sigma)$ define the rows and $t$ and $(\tau)$ the columns, is an invariant matrix of the matrix $[a_{st}]$ corresponding to the partition $(\lambda)$.*

Let $[a_{st}]$, $[b_{st}]$, and $[c_{st}]$ be three matrices of order $m^2$ such that

$$[c_{st}] = [a_{st}][b_{st}],$$

so that

$$c_{st} = \sum_r a_{sr} b_{rt}.$$

Let the products obtained from the matrices $[b_{st}]$ and $[c_{st}]$ corresponding to the product $P_{\sigma\tau}(S)$ from $[a_{st}]$ be $Q_{\sigma\tau}(S)$ and $R_{\sigma\tau}(S)$ respectively. It is required to prove that

$$\sum_{r,\mu,S,T} \phi_{rs}^{(\lambda)}(S^{-1})P_{\sigma\mu}(S)\phi_{tr}^{(\lambda)}(T^{-1})Q_{\mu\tau}(T) = \sum_S \phi_{ts}^{(\lambda)}(S^{-1})R_{\sigma\tau}(S). \tag{10.1;5}$$

On substituting $c_{st} = \sum a_{sr} b_{rt}$, it is easily seen that

$$R_{\sigma\tau}(S) = \sum_{\mu,T} P_{\sigma\mu}(T)Q_{\mu\tau}(T^{-1}S).$$

Hence it is sufficient to prove that

$$\sum \phi_{rs}^{(\lambda)}(S^{-1})P_{\sigma\mu}(S)\phi_{tr}^{(\lambda)}(T^{-1})Q_{\mu\tau}(T) = \sum \phi_{ts}^{(\lambda)}(S^{-1})P_{\sigma\mu}(T)Q_{\mu\tau}(T^{-1}S).$$

That is, on picking out the coefficient of $P_{\sigma\mu}(S)Q_{\mu\tau}(T)$, it is sufficient to prove that

$$\sum \phi_{rs}^{(\lambda)}(S^{-1})\phi_{tr}^{(\lambda)}(T^{-1}) = \sum \phi_{ts}^{(\lambda)}(T^{-1}S^{-1}).$$

Substituting for $S$, $T$, and $ST$ from (10.1; 4) in the equation

$$S.T = ST,$$

we obtain

$$\sum \phi_{ji}^{(\lambda)}(S^{-1})e_{ij}\,\phi_{rk}^{(\lambda)}(T^{-1})e_{kr} = \sum \phi_{ji}^{(\lambda)}(T^{-1}S^{-1})e_{ij}.$$

Picking out the coefficient of $e_{st}$, we obtain

$$\sum \phi_{rs}^{(\lambda)}(S^{-1})\phi_{tr}^{(\lambda)}(T^{-1}) = \phi_{ts}^{(\lambda)}(T^{-1}S^{-1}),$$

which proves the theorem.

EXAMPLE. We construct the invariant matrix of the matrix

$$A = \left\lfloor \begin{matrix} a_1, & a_2, \\ b_1, & b_2, \\ c_1, & c_2, \end{matrix} \right.$$

corresponding to the partition (2 1) of 3.

Consider the symmetric group of permutations on the three symbols $\alpha, \beta, \gamma$. There are two standard tableaux corresponding to (2 1), namely

$$T_1 = \begin{pmatrix} \alpha, & \beta \\ \gamma & \end{pmatrix}, \qquad T_2 = \begin{pmatrix} \alpha, & \gamma \\ \beta & \end{pmatrix},$$

from which we obtain

$$\begin{aligned} 3e_{11} &= 1+(\alpha\beta)-(\alpha\gamma)-(\alpha\beta\gamma), \\ 3e_{22} &= 1+(\alpha\gamma)-(\alpha\beta)-(\alpha\gamma\beta), \\ 3e_{12} &= (\beta\gamma)+(\alpha\gamma\beta)-(\alpha\beta\gamma)-(\alpha\gamma), \\ 3e_{21} &= (\beta\gamma)+(\alpha\beta\gamma)-(\alpha\gamma\beta)-(\alpha\beta): \end{aligned}$$

Hence the invariant matrix of order $14^2$ is given by

$$T(A) \quad \left[ \begin{array}{llllll}
a_1b_2c_3+a_2b_1c_3-a_3b_2c_1-a_2b_3c_1, & a_1b_3c_2+a_3b_1c_2-a_2b_3c_1-a_3b_2c_1, & 2a_1b_1c_2-a_2b_1c_1-a_1b_2c_1, & 0, & 2a_1b_1c_3-a_3b_1c_1-a_1b_3c_1, & 0, \\
a_1b_3c_2+a_2b_3c_1-a_3b_1c_2-a_2b_1c_3, & a_1b_2c_3+a_3b_2c_1-a_2b_1c_3-a_3b_1c_2, & 2a_1b_2c_1-a_2b_1c_1-a_1b_1c_2, & 0, & 2a_1b_3c_1-a_3b_1c_1-a_1b_1c_3, & 0, \\
a_1a_2b_3-a_2a_3b_1, & a_1a_3b_2-a_2a_3b_1, & a_1^2b_2-a_1a_2b_1, & 0, & a_1^2b_3-a_1a_3b_1, & 0, \\
-\frac{1}{2}(a_1a_2b_3-a_2a_3b_1), & -\frac{1}{2}(a_1a_3b_2-a_2a_3b_1), & -\frac{1}{2}(a_1^2b_2-a_1a_2b_1), & 0, & -\frac{1}{2}(a_1^2b_3-a_1a_3b_1), & 0, \\
a_1a_2c_3-a_2a_3c_1, & a_1a_3c_2-a_2a_3c_1, & a_1^2c_2-a_1a_2c_1, & 0, & a_1^2c_3-a_1a_3c_1, & 0, \\
-\frac{1}{2}(a_1a_2c_3-a_2a_3c_1), & -\frac{1}{2}(a_1a_3c_2-a_2a_3c_1), & -\frac{1}{2}(a_1^2c_2-a_1a_2c_1), & 0, & -\frac{1}{2}(a_1^2c_3-a_1a_3c_1), & 0, \\
b_1b_2c_3-b_2b_3c_1, & b_1b_3c_2-b_2b_3c_1, & b_1^2c_2-b_1b_2c_1, & 0, & b_1^2c_3-b_1b_3c_1, & 0, \\
-\frac{1}{2}(b_1b_2c_3-b_2b_3c_1), & -\frac{1}{2}(b_1b_3c_2-b_2b_3c_1), & -\frac{1}{2}(b_1^2c_2-b_1b_2c_1), & 0, & -\frac{1}{2}(b_1^2c_3-b_1b_3c_1), & 0, \\
\frac{1}{2}(a_1b_2b_3-a_3b_1b_2), & \frac{1}{2}(a_1b_2b_3-a_2b_1b_3), & \frac{1}{2}(a_1b_1b_2-a_2b_1^2), & 0, & \frac{1}{2}(a_1b_1b_3-a_3b_1^2), & 0, \\
\frac{1}{2}(a_1b_2b_3-a_3b_1b_2), & \frac{1}{2}(a_1b_2b_3-a_2b_1b_3), & \frac{1}{2}(a_1b_1b_2-a_2b_1^2), & 0, & \frac{1}{2}(a_1b_1b_3-a_3b_1^2), & 0, \\
\frac{1}{2}(a_1c_2c_3-a_3c_1c_2), & \frac{1}{2}(a_1c_2c_3-a_2c_1c_3), & \frac{1}{2}(a_1c_1c_2-a_2c_1^2), & 0, & \frac{1}{2}(a_1c_1c_3-a_3c_1^2), & 0, \\
\frac{1}{2}(a_1c_2c_3-a_3c_1c_2), & \frac{1}{2}(a_1c_2c_3-a_2c_1c_3), & \frac{1}{2}(a_1c_1c_2-a_2c_1^2), & 0, & \frac{1}{2}(a_1c_1c_3-a_3c_1^2), & 0, \\
\frac{1}{2}(b_1c_2c_3-b_3c_1c_2), & \frac{1}{2}(b_1c_2c_3-b_2c_1c_3), & \frac{1}{2}(b_1c_1c_2-b_2c_1^2), & 0, & \frac{1}{2}(b_1c_1c_3-b_3c_1^2), & 0, \\
\frac{1}{2}(b_1c_2c_3-b_3c_1c_2), & \frac{1}{2}(b_1c_2c_3-b_2c_1c_3), & \frac{1}{2}(b_1c_1c_2-b_2c_1^2), & 0, & \frac{1}{2}(b_1c_1c_3-b_3c_1^2), & 0,
\end{array} \right.$$

$$\left.\begin{array}{lllllll}
2a_2 b_2 c_3 & 0, & a_1 b_2 c_2 + a_2 b_1 c_2 & *, & a_1 b_3 c_3 + a_3 b_1 c_3 & a_2 b_3 c_3 + a_3 b_2 c_3 & * \\
\quad - a_3 b_2 c_2 - a_2 b_3 c_2, & & \qquad -2a_2 b_2 c_1, & & \qquad -2a_3 b_3 c_2, & \qquad -2a_3 b_2 c_1; & \\
2a_2 b_3 c_2 & 0, & a_1 b_2 c_2 + a_2 b_2 c_1 & *, & a_1 b_3 c_3 + a_3 b_3 c_1 & , a_2 b_3 c_3 + a_3 b_3 c_2 & * \\
\quad - a_3 b_2 c_2 - a_2 b_2 c_3, & & \qquad -2a_2 b_1 c_2, & & \qquad -2a_3 b_1 c_3, & \qquad -2a_3 b_2 c_3 & \\
a_2^2 b_3 - a_2 a_3 b_2, & 0, & a_1 a_2 b_2 - a_2^2 b_1, & *, & a_1 a_3 b_3 - a_3^2 b_1, & a_2 a_3 b_3 - a_3^2 b_2, & * \\
-\frac{1}{2}(a_2^2 b_3 - a_2 a_3 b_2), & 0, & -\frac{1}{2}(a_1 a_2 b_2 - a_2^2 b_1), & *, & -\frac{1}{2}(a_1 a_3 b_3 - a_3^2 b_1), & -\frac{1}{2}(a_2 a_3 b_3 - a_3^2 b_2), & * \\
a_2^2 c_3 - a_2 a_3 c_2, & 0, & a_1 a_2 c_2 - a_2^2 c_1, & *, & a_1 a_3 c_3 - a_3^2 c_1, & a_2 a_3 c_3 - a_3^2 c_2, & * \\
-\frac{1}{2}(a_2^2 c_3 - a_2 a_3 c_2), & 0, & -\frac{1}{2}(a_1 a_2 c_2 - a_2^2 c_1), & *, & -\frac{1}{2}(a_1 a_3 c_3 - a_3^2 c_1), & -\frac{1}{2}(a_2 a_3 c_3 - a_3^2 c_2), & * \\
b_2^2 c_3 - b_2 b_3 c_2, & 0, & b_1 b_2 c_2 - b_2^2 c_1, & *, & b_1 b_3 c_3 - b_3^2 c_1, & b_2 b_3 c_3 - b_3^2 c_2, & * \\
-\frac{1}{2}(b_2^2 c_3 - b_2 b_3 c_2), & 0, & -\frac{1}{2}(b_1 b_2 c_2 - b_2^2 c_1), & *, & -\frac{1}{2}(b_1 b_3 c_3 - b_3^2 c_1), & -\frac{1}{2}(b_2 b_3 c_3 - b_3^2 c_2), & * \\
\frac{1}{2}(a_2 b_2 b_3 - a_3 b_2^2), & 0, & \frac{1}{2}(a_1 b_2^2 - a_2 b_1 b_2), & *, & \frac{1}{2}(a_1 b_3^2 - a_3 b_1 b_3), & \frac{1}{2}(a_2 b_3^2 - a_3 b_2 b_3), & * \\
\frac{1}{2}(a_2 b_2 b_3 - a_3 b_2^2), & 0, & \frac{1}{2}(a_1 b_2^2 - a_2 b_1 b_2), & *, & \frac{1}{2}(a_1 b_3^2 - a_3 b_1 b_3), & \frac{1}{2}(a_2 b_3^2 - a_3 b_2 b_3), & * \\
\frac{1}{2}(a_2 c_2 c_3 - a_3 c_2^2), & 0, & \frac{1}{2}(a_1 c_2^2 - a_2 c_1 c_2), & *, & \frac{1}{2}(a_1 c_3^2 - a_3 c_1 c_3), & \frac{1}{2}(a_2 c_3^2 - a_3 c_2 c_3), & * \\
\frac{1}{2}(a_2 c_2 c_3 - a_3 c_2^2), & 0, & \frac{1}{2}(a_1 c_2^2 - a_2 c_1 c_2), & *, & \frac{1}{2}(a_1 c_3^2 - a_3 c_1 c_3), & \frac{1}{2}(a_2 c_3^2 - a_3 c_2 c_3), & * \\
\frac{1}{2}(b_2 c_2 c_3 - b_3 c_2^2), & 0, & \frac{1}{2}(b_1 c_2^2 - b_2 c_1 c_2), & *, & \frac{1}{2}(b_1 c_3^2 - b_3 c_1 c_3), & \frac{1}{2}(b_2 c_3^2 - b_3 c_2 c_3), & * \\
\frac{1}{2}(b_2 c_2 c_3 - b_3 c_2^2), & 0, & \frac{1}{2}(b_1 c_2^2 - b_2 c_1 c_2), & *, & \frac{1}{2}(b_1 c_3^2 - b_3 c_1 c_3), & \frac{1}{2}(b_2 c_3^2 - b_3 c_2 c_3), & *
\end{array}\right]$$

The asterisks indicate columns each identical with the preceding column. It will be seen that the invariant matrix so formed is singular, having more than the necessary number of rows and columns. Some columns consist of zeros, others are repeated. However, if the columns of zeros and the corresponding rows are omitted, and if each repeated column is written once only, and the corresponding rows added, we obtain the following invariant matrix, which is non-singular.

$$\left[\begin{array}{llll}
a_1 b_2 c_3 + a_2 b_1 c_3 & a_1 b_3 c_2 + a_3 b_1 c_2 & 2a_1 b_1 c_2 & 2a_1 b_1 c_3 \\
\quad - a_3 b_2 c_1 - a_2 b_3 c_1, & \quad - a_2 b_3 c_1 - a_3 b_2 c_1, & \quad - a_1 b_2 c_1 - a_2 b_1 c_1, & \quad - a_1 b_3 c_1 - a_3 b_1 c_1, \\
a_1 b_3 c_2 + a_2 b_3 c_1 & a_1 b_2 c_3 + a_3 b_2 c_1 & 2a_1 b_2 c_1 & 2a_1 b_3 c_1 \\
\quad - a_3 b_1 c_2 - a_2 b_1 c_3, & \quad - a_2 b_1 c_3 - a_3 b_1 c_2, & \quad - a_1 b_1 c_2 - a_2 b_1 c_1, & \quad - a_1 b_1 c_3 - a_3 b_1 c_1, \\
a_1 a_2 b_3 - a_2 a_3 b_1, & a_1 a_3 b_2 - a_2 a_3 b_1, & a_1^2 b_2 - a_1 a_2 b_1, & a_1^2 b_3 - a_1 a_3 b_1, \\
a_1 a_2 c_3 - a_2 a_3 c_1, & a_1 a_3 c_2 - a_2 a_3 c_1, & a_1^2 c_2 - a_1 a_2 c_1, & a_1^2 c_3 - a_1 a_3 c_1, \\
b_1 b_2 c_3 - b_2 b_3 c_1, & b_1 b_3 c_2 - b_2 b_3 c_1, & b_1^2 c_2 - b_1 b_2 c_1, & b_1^2 c_3 - b_1 b_3 c_1, \\
a_1 b_2 b_3 - a_3 b_1 b_2, & a_1 b_2 b_3 - a_2 b_1 b_3, & a_1 b_1 b_2 - a_2 b_1^2, & a_1 b_1 b_3 - a_3 b_1^2, \\
a_1 c_2 c_3 - a_3 c_1 c_2, & a_1 c_2 c_3 - a_2 c_1 c_3, & a_1 c_1 c_2 - a_2 c_1^2, & a_1 c_1 c_3 - a_3 c_1^2, \\
b_1 c_2 c_3 - b_3 c_1 c_2, & b_1 c_2 c_3 - b_2 c_1 c_3, & b_1 c_1 c_2 - b_2 c_1^2, & b_1 c_1 c_3 - b_3 c_1^2,
\end{array}\right.$$

$$\left.\begin{array}{llll}
2a_2 b_2 c_3 & a_1 b_2 c_2 + a_2 b_1 c_2 & a_1 b_3 c_3 + a_3 b_1 c_3 & a_2 b_3 c_3 + a_3 b_2 c_3 \\
\quad - a_2 b_3 c_2 - a_3 b_2 c_2, & \qquad -2a_2 b_2 c_1, & \qquad -2a_3 b_3 c_1, & \qquad -2a_3 b_3 c_2 \\
2a_2 b_3 c_2 & a_1 b_2 c_2 + a_2 b_2 c_1 & a_1 b_3 c_3 + a_3 b_3 c_1 & a_2 b_3 c_3 + a_3 b_3 c_2 \\
\quad - a_2 b_2 c_3 - a_3 b_2 c_2, & \qquad -2a_2 b_1 c_2, & \qquad -2a_3 b_1 c_3, & \qquad -2a_3 b_2 c_3 \\
a_2^2 b_3 - a_2 a_3 b_2, & a_1 a_2 b_2 - a_2^2 b_1, & a_1 a_3 b_3 - a_3^2 b_1, & a_2 a_3 b_3 - a_3^2 b_2 \\
a_2^2 c_3 - a_2 a_3 c_2, & a_1 a_2 c_2 - a_2^2 c_1, & a_1 a_3 c_3 - a_3^2 c_1, & a_2 a_3 c_3 - a_3^2 c_2 \\
b_2^2 c_3 - b_2 b_3 c_2, & b_1 b_2 c_2 - b_2^2 c_1, & b_1 b_3 c_3 - b_3^2 c_1, & b_2 b_3 c_3 - b_3^2 c_2 \\
a_2 b_2 b_3 - a_3 b_2^2, & a_1 b_2^2 - a_2 b_1 b_2, & a_1 b_3^2 - a_3 b_1 b_3, & a_2 b_3^2 - a_3 b_2 b_3 \\
a_2 c_2 c_3 - a_3 c_2^2, & a_1 c_2^2 - a_2 c_1 c_2, & a_1 c_3^2 - a_3 c_1 c_3, & a_2 c_3^2 - a_3 c_2 c_3 \\
b_2 c_2 c_3 - b_3 c_2^2, & b_1 c_2^2 - b_2 c_1 c_2, & b_1 c_3^2 - b_3 c_1 c_3, & b_2 c_3^2 - b_3 c_2 c_3
\end{array}\right] \tag{10.1; 6}$$

It may be verified that for either matrix $T(A).T(B) = T(AB)$.

V. *If in the invariant matrix of Theorem IV, columns of zeros and the corresponding rows are omitted, and repeated columns are written once only, and the corresponding rows added, we obtain a non-singular invariant matrix.*

It is obvious that the matrix is still invariant, for the omission of columns of zeros and corresponding rows cannot affect its invariancy, and an obvious transformation, if a column is repeated, will reduce all except the first column to zeros, and add the corresponding rows. It is only necessary to show that the resulting matrix is non-singular. This we shall do by considering the invariant matrix of a matrix in diagonal form.

Let $(\tau) = (1^a 2^b 3^c ...)$. Use the symbols $\alpha_1, \alpha_2, ..., \alpha_a, \beta_1, \beta_2, ..., \beta_b, \gamma_1, ..., \gamma_c, ...$, etc., in this order, for the construction of the Young tableaux. Now for any tableau $T_i$ in which there are two $\alpha$'s in the same column, say $\alpha_j$ and $\alpha_k$, the product $N_i$ will admit of a factor $1-(\alpha_j \alpha_k)$ on the right. Since the substitution $(\alpha_j \alpha_k)$ on the extreme right of a substitutional expression is equivalent to the interchange of the identical columns corresponding to $\alpha_j$ and $\alpha_k$, any expression with a factor $1-(\alpha_j \alpha_k)$ on the right will give a zero term in the invariant matrix. In particular, to $M_s P_s \sigma_{si} N_i$ there corresponds a zero term for all $s$, and the complete column of the invariant matrix corresponding to the tableau $T_i$ is zero. The same result holds if two $\beta$'s or $\gamma$'s, etc., are in the same column.

Next consider a tableau $T_i$ and any other (standard) tableau $T_j$ which can be obtained from it by rearrangement of the $\alpha$'s, $\beta$'s, $\gamma$'s, etc., among themselves. A factor $\sigma_{ij}$ operating on the extreme right of any substitutional expression will have the effect of interchanging sets of identical columns. Hence $M_k P_k \sigma_{ki} N_i$ will give an identical expression in the invariant matrix to $M_k P_k \sigma_{ki} N_i \sigma_{ij} = M_k P_k \sigma_{kj} N_j$. Hence the two columns of the invariant matrix corresponding to $T_i$ and $T_j$ are identical.

It remains to show that if we choose one tableau only from each arrangement of the $\alpha$'s, $\beta$'s, etc., irrespective of suffixes, omitting those with two $\alpha$'s or two $\beta$'s, etc., in the same column, we obtain a non-singular invariant matrix.

Let the original matrix be the diagonal matrix

$$\Omega = \operatorname{diag}(\omega_1, \omega_2, ..., \omega_m).$$

Clearly, to get a non-zero term of the invariant matrix we must have $(\sigma) = (\tau)$, or at least one term in each product $P_S$ will have different suffixes and the expression will be zero.

For $(\sigma) = (\tau) = (1^a 2^b 3^c \ldots)$ there will be $a!\,b!\,c!\ldots$ products $P_S$ which are non-zero, corresponding to the terms of the product of the symmetric groups on the $\alpha$'s, $\beta$'s, $\gamma$'s,... respectively. Each product $P_S$ will be equal to $\omega_1^a \omega_2^b \omega_3^c \ldots$.

Clearly $M_i P_i N_i$ will correspond to a multiple of $\omega_1^a \omega_2^b \omega_3^c \ldots$ with positive coefficient, for the only terms that will signify will correspond to partial products in which the identity is taken from $M_i$ and from $N_i$, and only such expressions are taken from $P_i$ as permute the $\alpha$'s, $\beta$'s, $\gamma$'s, etc., among themselves. For any other substitutional expression obtained from $M_i P_i N_i$ will permute symbols corresponding to different letters, and will give a zero result. Again, it is clear that if $\sigma_{ij}$ permutes the letters among the $\alpha$'s, $\beta$'s, $\gamma$'s,..., then there is no expression in the product

$$M_i P_i \sigma_{ij} N_j$$

which will not give a zero result. The only case in which a non-zero result could possibly be obtained is when $\sigma_{ik}$ belongs to $P_i N_k$ and $\sigma_{jk}$ permutes the $\alpha$'s, $\beta$'s, etc., among themselves only, and investigation shows that the definition of $M_i$ is designed to make the result zero in this case also.

Thus it will be seen that the resulting invariant matrix is diagonal, every leading diagonal term differs from zero, and the matrix is non-singular.

We show next that the character of this invariant matrix is equal to the corresponding $S$-function of the characteristic roots of $A$.

The spur of $T(A)$ is clearly

$$\sum_{S,s,\sigma} \phi_{ss}^{(\lambda)}(S^{-1}) P_{\sigma\sigma}(S).$$

But $\sum_s \phi_{ss}^{(\lambda)}(S^{-1})$ is the spur of the matrix representation of $S^{-1}$, namely $\chi^{(\lambda)}(S)$, so that the spur of $T(A)$ is

$$\begin{aligned}\sum_{S,\sigma} \chi^{(\lambda)}(S) P_{\sigma\sigma}(S) &= \sum_{\sigma} |A_{\sigma\sigma}|^{(\lambda)} \\ &= \{\lambda\},\end{aligned}$$

the $S$-function of the characteristic roots of $A$.

Before proving that this invariant matrix is irreducible we prove some theorems concerning the order of the matrix.

Since $T(I)$ is a unit matrix, the degree of $T(A)$ is equal to the spur of $T(I)$ which is the $S$-function $\{\lambda\}$ of the roots of $(1-x)^m$. This we have shown to be (§ 7.2, Theorem III)

$\phi_m^{(\lambda)} = f^{(\lambda)}/n!$ times the product of the first $\lambda_i$ terms from each $i$th row of

$$\begin{array}{lllll} m, & m+1, & m+2, & m+3, & \ldots \\ m-1, & m, & m+1, & m+2, & \ldots \\ m-2, & m-1, & m, & m+1, & \ldots \end{array} \qquad (10.1;7)$$

where $f^{(\lambda)}$ is the degree of the corresponding character of the symmetric group.

Hence we find an expression for $\phi_m^{(\lambda)}$ by counting the number of Young tableaux used in Theorem V.

VI. *The expression $\phi_m^{(\lambda)}$ gives the number of standard Young tableaux corresponding to a partition $(\lambda)$ of $n$, that can be formed from any $n$ of a set of $m$ symbols, in which repetitions of the same symbol are allowed, but not in the same column.*

If $\alpha_1,\ldots, \alpha_m$ are the $m$ symbols, take from each tableau, as defined in Theorem VI, that part which consists entirely of $\alpha_1$'s; $\alpha_1$'s and $\alpha_2$'s; $\alpha_1$'s, $\alpha_2$'s, and $\alpha_3$'s; and so on. Each part corresponds to a definite partition. We obtain a sequence of $m$ partitions such that

(1) zero is taken to be a partition,

(2) each partition is obtained from the previous partition by adding to any of the parts or leaving them unaltered,

(3) the first partition is either zero or contains one part,

(4) the $(i+1)$th part of the $(r+1)$th partition is less than or equal to the $i$th part of the $r$th partition,

(5) the $m$th partition is $(\lambda)$.

VII. *The number of sequences of partitions satisfying the above five rules is $\phi_m^{(\lambda)}$.*

As an example we consider $\phi_3^{(21)}$. We have, from (10.1;7),

$$\phi_3^{(21)} = \tfrac{2}{6}.3.4.2 = 8.$$

as may be verified from the order of the invariant matrix already given as an example. The eight Young tableaux of Theorem III are

$$\begin{pmatrix}\alpha, & \alpha\\ \beta & \end{pmatrix},\quad \begin{pmatrix}\alpha, & \alpha\\ \gamma & \end{pmatrix},\quad \begin{pmatrix}\beta, & \beta\\ \gamma & \end{pmatrix},$$
$$\begin{pmatrix}\alpha, & \beta\\ \beta & \end{pmatrix},\quad \begin{pmatrix}\alpha, & \gamma\\ \gamma & \end{pmatrix},\quad \begin{pmatrix}\beta, & \gamma\\ \gamma & \end{pmatrix},$$
$$\begin{pmatrix}\alpha, & \beta\\ \gamma & \end{pmatrix},\quad \begin{pmatrix}\alpha, & \gamma\\ \beta & \end{pmatrix}.$$

The eight sequences of partitions are

$$(2),\ (2\,1),\ (2\,1);\quad (2),\ (2),\ (2\,1);\quad (0),\ (2),\ (2\,1);$$
$$(1),\ (2\,1),\ (2\,1);\quad (1),\ (1),\ (2\,1);\quad (0),\ (1),\ (2\,1);$$
$$(1),\ (2),\ (2\,1);\quad (1),\ (1^2),\ (2\,1).$$

Now each partition in any sequence is one of the terms obtained with coefficient unity, on multiplying the previous partition by $h_r$ for some $r$.

Consider the expansion of

$$[f(x)]^m = [1+h_1x+h_2x^2+h_3x^3+\ldots]^m,$$

and let the coefficient of $x^n$ be expressed as a sum of $S$-functions. We pick out the coefficient of $\{\lambda\}x^n$. Each of the terms $\{\lambda\}x^n$ is obtained from a partial product

$$h_{a_1}x^{a_1}.h_{a_2}x^{a_2}.h_{a_3}x^{a_3}\ldots h_{a_m}x^{a_m}.$$

On multiplying, we get for each term $\{\lambda\}x^n$ a sequence of partitions as defined in Theorem VI, and conversely.

VIII. *If* $\{\lambda\}$ *is an S-function associated with the series*

$$f(x) = 1+h_1x+h_2x^2+h_3x^3+\ldots,$$

*then* $$[f(x)]^m = 1+\sum \phi_m^{(\lambda)}\{\lambda\}x^n$$

*summed for all partitions* $(\lambda)$ *of every integer* $n$.

This is a somewhat remarkable result which, since it is true for every positive integral value of $m$, and since the coefficients can be shown, by the use of the binomial theorem, to be polynomials in the index, must clearly hold for fractional and negative indices.

A more direct proof seems indicated, and we give it here.

It is known that

$$f(\rho x_1)f(\rho x_2)\ldots f(\rho x_m) = 1+\sum\{\lambda\}\{x;\lambda\}\rho^n,$$

where $\{x;\lambda\}$ denotes an $S$-function of the quantities $x_1,\ldots, x_m$.

Now, if $$x_1 = x_2 = \dots = x_m = 1,$$
$\{x;\lambda\}$ becomes an $S$-function of $(1-x)^{-m}$, which is $\phi_m^{(\lambda)}$.

Hence $$[f(\rho)]^m = 1+\sum \phi_m^{(\lambda)}\{\lambda\}\rho^n,$$
which proves the theorem.

The same reasoning by which we deduced Theorem VIII from Theorem V, but reversed, will serve to deduce consecutively Theorems VII and VI from Theorem VIII. Assuming Theorem VIproved by the second method, and Theorems VII and VI deduced from it, an alternative proof of Theorem V would be to show from Theorem VI that the order of the invariant matrix is the correct order for invariant matrices corresponding to this partition, so that it must be non-singular.

Another theorem may be deduced from Theorem V by using the fact that the spur of an invariant matrix is an $S$-function of the characteristic roots of the original matrix.

Take the original matrix as $X = \text{diag}(x_1,\dots,x_m)$, and let $\{\lambda\}$ denote an $S$-function of $x_1,\dots, x_m$. Let

$$\{\lambda\} = \sum K_{\lambda\mu}\, x_1^{\mu_1} x_2^{\mu_2}\dots.$$

By counting the terms in the leading diagonal of $T(X)$ which correspond to the product $x_1^{\mu_1}x_2^{\mu_2}\dots$, we have

IX. *If* $$\{\lambda\} = \sum K_{\lambda\mu}\, x_1^{\mu_1} x_2^{\mu_2} x_3^{\mu_3}\dots,$$
*then* $K_{\lambda\mu}$ *is the number of standard Young tableaux that can be formed corresponding to the partition* $(\lambda)$ *from the symbol* $x_1$ *used* $\mu_1$ *times, the symbol* $x_2$ *used* $\mu_2$ *times, and so on, no symbol being repeated in the same column.*

As an example,

$$\{3\,1\} = \sum x_1^3 x_2 + \sum x_1^2 x_2^2 + 2\sum x_1^2 x_2 x_3 + 3\sum x_1 x_2 x_3 x_4,$$

and the corresponding tableaux are

$$\begin{pmatrix} x_1, & x_1, & x_1 \\ x_2 & & \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_1, & x_2 \\ x_2 & & \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_1, & x_2 \\ x_3 & & \end{pmatrix},$$

$$\begin{pmatrix} x_1, & x_1, & x_3 \\ x_2 & & \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_2, & x_3 \\ x_4 & & \end{pmatrix}, \quad \begin{pmatrix} x_1, & x_2, & x_4 \\ x_3 & & \end{pmatrix},$$

$$\begin{pmatrix} x_1, & x_3, & x_4 \\ x_2 & & \end{pmatrix}.$$

The coefficient $K_{\lambda\mu}$ has been shown (6.4; 6) to be the same as the coefficient in the expansion

$$h_{\mu_1} h_{\mu_2} \dots h_{\mu} = \sum K_{\lambda\mu}\{\lambda\}.$$

We now complete the proof of Theorem III by showing that the invariant matrix is irreducible. This will follow if it is shown that the elements of $T(A)$, considered as functions of the elements of $A$, are linearly independent.

Two elements of $T(A)$, namely

$$\sum \phi_{ts}^{(\lambda)}(S^{-1})P_{\sigma\tau}(S), \qquad \sum \phi_{t's'}^{(\lambda)}(S^{-1})P_{\sigma'\tau'}(S)$$

will be of different weight in the elements of some row or column of $A$, unless

$$(\sigma) = (\sigma') \quad \text{and} \quad (\tau) = (\tau').$$

Hence, if any linear relation exists, there will be a linear relation between the elements for some fixed $(\sigma)$ and fixed $(\tau)$.

Let $A_{\sigma\tau}$ be an $n$-rowed minor of $A$. If $(\sigma)$ and $(\tau)$ contain no repetitions, then the $n!$ products $P_{\sigma\tau}(S)$ are linearly independent. From the theory of matrix representations of the symmetric group, the $f^{(\lambda)2}$ terms

$$\sum \phi_{ts}^{(\lambda)}(S^{-1})P_{\sigma\tau}(S)$$

are linearly independent also.

If, however, there are repetitions in the sets $(\sigma)$ and $(\tau)$, and

$$(\sigma) \equiv (1^{\sigma_1}, 2^{\sigma_2}, \dots),$$
$$(\tau) \equiv (1^{\tau_1}, 2^{\tau_2}, \dots),$$

then linear relations exist between the terms $P_{\sigma\tau}(S)$. This linear dependency is fully described by the fact that if $S$ is multiplied on the right by any operation of the symmetric group on the first set of $\sigma_1$ symbols, or the next set of $\sigma_2$ symbols, etc., or again if $S$ is multiplied on the left by any operation of the symmetric group on the first set of $\tau_1$ symbols, or the next set of $\tau_2$ symbols, etc., then $P_{\sigma\tau}(S)$ is unaltered.

The corresponding elements of the Frobenius algebra would be made identical if the corresponding expressions were multiplied on the right by the product of the symmetric groups on the respective sets of $\sigma_1$, $\sigma_2$,... symbols, and on the left by the product of the symmetric groups on the respective sets of $\tau_1$, $\tau_2$,... symbols. Denote these symmetric groups by $P_{\sigma_1}$, $P_{\sigma_2}$,..., $P_{\tau_1}$, $P_{\tau_2}$,... respectively.

The number of linearly independent terms

$$\sum \phi_{ts}^{(\lambda)}(S^{-1})P_{\sigma\tau}(S)$$

is thus equal to the number of linearly independent terms in the matrix elements of

$$P_{\tau_1} P_{\tau_2} \dots P_{\tau_n} G P_{\sigma_1} P_{\sigma_2} \dots P_{\sigma_n},$$

where $G$ is the group matrix corresponding to the partition $(\lambda)$.

Now $P_{\tau_1} P_{\tau_2} \dots P_{\tau_n}$ is a multiple of a characteristic unit with compound character corresponding to

$$h_{\tau_1} h_{\tau_2} \dots h_{\tau_n},$$

and the rank of the matrix representation in the sub-algebra corresponding to the partition $(\lambda)$ is the coefficient $K_{\lambda\tau}$ of $\{\lambda\}$ in the expansion

$$h_{\tau_1} h_{\tau_2} \dots h_{\tau_n} = \sum K_{\lambda\tau}\{\lambda\}.$$

Similarly, the rank of the matrix representation of $P_{\sigma_1} P_{\sigma_2} \dots P_{\sigma_n}$ is $K_{\lambda\sigma}$. Hence the required number of linearly independent terms is $K_{\lambda\sigma} K_{\lambda\tau}$.

But from Theorem IX, $K_{\lambda\sigma}$, $K_{\lambda\tau}$ are the number of rows and columns of the invariant matrix which correspond to the sets $(\sigma)$, $(\tau)$ respectively. Hence the corresponding $K_{\lambda\sigma} K_{\lambda\tau}$ elements of the invariant matrix are linearly independent, and the invariant matrix is irreducible.

## 10.2. The classical canonical form of an invariant matrix

If the characteristic roots of a matrix $A$ are all distinct, $A$ may be transformed into diagonal form

$$A = \operatorname{diag}(\omega_1, \omega_2, \dots, \omega_n).$$

Clearly the invariant matrices defined in Theorem V will be diagonal also, and will represent the classical canonical form of the invariant matrix.

Each diagonal term must clearly be a product of powers of the roots, a fact which follows from the equation

$$\operatorname{diag}(\omega_1, \omega_2, \dots, \omega_n)\operatorname{diag}(\omega_1', \omega_2', \dots, \omega_n') = \operatorname{diag}(\omega_1 \omega_1', \omega_2 \omega_2', \dots, \omega_n \omega_n').$$

But the spur of $A^{[\lambda]}$ is $\{\lambda\}$. Hence, if $\{\lambda\}$ is expressed as a sum of products of powers of the roots, each term will be a diagonal term in the canonical form of $A^{[\lambda]}$ (4).

This result holds also for equal characteristic roots of $A$, provided that the reduced characteristic equation has no multiple roots, so that the canonical form of $A$ is diagonal.

The case of matrices with repeated roots of the reduced equation, so that the canonical form is not diagonal, presents more difficulty (5).

Henceforward we use the following nomenclature. Since we are dealing strictly with the classical canonical form of a matrix, we shall assume every matrix to be reduced to this form.

We shall use $\dotplus$ to denote the direct sum of two matrices, i.e.

$$A \dotplus B = \begin{array}{cc} A & 0 \\ \cdots & \cdots \\ 0 & B \end{array}$$

$\dot{-}$ for a corresponding difference, and **x** to denote the direct product of two matrices. Thus, if $A = [a_{st}]$ and $B = [b_{st}]$ are matrices of order $n^2$ and $m^2$ respectively, $A \times B$ denotes the matrix of order $n^2m^2$ with elements $a_{st}b_{s't'}$, the two suffixes $s$ and $s'$ defining the rows, and $t$ and $t'$ the columns. Scalar multiplication by $n$ denotes the direct sum of the same matrix repeated $n$ times.

The matrix sometimes denoted by $C_n(\mu)$, namely with $n$ terms in the leading diagonal each equal to $\mu$, and $n-1$ terms each equal to unity in the first diagonal above, will be denoted by $[n]_\mu$.

Hence, e.g. $[3]_\mu \dotplus [2]_0 \dotplus [1]_\nu$ will denote the matrix

$$\begin{bmatrix} \mu, & 1, & & & & \\ & \mu, & 1, & & & \\ & & \mu, & & & \\ & & & 0, & 1, & \\ & & & & 0, & \\ & & & & & 1 \end{bmatrix}$$

Taking account of canonical form only, addition and multiplication are clearly commutative, and multiplication is distributive with respect to addition.

The direct product $[n]_\mu \times [m]_\nu$ may be written as

$$A = [a_{ss',tt'}],$$

where

$$a_{pq,pq} = \mu\nu,$$

$$a_{pq,p+1.q} = \nu,$$

$$a_{pq,p.q+1} = \mu,$$

$$a_{pq,p+1.q+1} = 1,$$

and all other terms of the matrix are zero.

Let the canonical form of this matrix be

$$\sum_{i=1}^{r} k_i[i]_{\mu\nu}$$

In order to find the maximum degree $r$ of a submatrix, and the number of times $k_i$ a submatrix of degree $i$ is repeated, we use the method of chains (6).

Of the $mn$ pairs of numbers $(p,q)$ which define the rows and columns of the matrix we form a chain of $j$ such pairs

$$(p_1,q_1),\quad (p_2,q_2),\quad \ldots,\quad (p_j,q_j)$$

such that

$$a_{p_i q_i, p_{i+1} q_{i+1}} \neq 0.$$

The number of terms in the longest chain that can be so formed is the degree $r$ of the largest submatrix. For, corresponding to the given chain of $r$ terms there will be a non-zero term in the matrix $(A-\mu\nu I)^{r-1}$, $I$ denoting the unit matrix, and conversely, for any non-zero term in the matrix $(A-\mu\nu I)^r$ it would be possible to form a chain of $(r+1)$ terms.

Further,† $k_r$ is the rank of the matrix $(A-\mu\nu I)^{r-1}$, and is thus equal to the maximum number of chains of $r$ pairs (all distinct) which can be formed from the $mn$ pairs $(p,q)$.

Again, by considering the ranks of the matrices $(A-\mu\nu I)^{r-2}$, $(A-\mu\nu I)^{r-3}$, etc., it follows that $k_{r-1}$ is the number of chains of $(r-1)$ pairs that can be formed from the remainder of the $mn$ pairs, $k_{r-2}$ is the number of further chains of $(r-2)$ pairs, and so on.

Now the $mn$ pairs can be arranged in a rectangle

$$\begin{array}{ccccc}
(1,1), & (1,2), & (1,3), & \ldots, & (1,n),\\
(2,1), & (2,2), & (2,3), & \ldots, & (2,n),\\
 & & & & \\
(m,1), & (m,2), & (m,3), & \ldots, & (m,n).
\end{array}$$

If $\mu \neq 0$, $\nu \neq 0$, corresponding to any term in a chain, the next term is either immediately to the right, immediately below, or diagonally to the right and below. For maximum length of chains the last case may be omitted.

Clearly the longest chain may consist of the first row and last column. The next longest may consist of the remaining terms in the second row and $(n-1)$th column; the next of the remaining terms in the third row and $(n-2)$th column, and so on. Hence

I.

$$[n]_\mu \times [m]_\nu = [n+m-1]_{\mu\nu} \dotplus [n+m-3]_{\mu\nu} \dotplus$$
$$\dotplus [n+m-5]_{\mu\nu} \dotplus \ldots \dotplus [n-m+1]_{\mu\nu},$$

*for* $\mu \neq 0$, $\nu \neq 0$, $n \geqslant m$.

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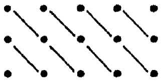
† See § 1.7, last section.

For the case of zero roots of the characteristic equation, if $\mu = 0$, $\nu = 0$, we have
$$a_{pq,p+1\,q+1} = 1,$$
and all other terms of the matrix $A$ are zero.

The chains are formed by connecting each pair in the rectangle with that diagonally to the right and below. The result may be illustrated with nodes replacing the pairs $(p, q)$ for the case $n = 5$, $m = 3$.



Clearly we obtain

II.
$$[n]_0 \times [m]_0 = 2[1]_0 \dotplus 2[2]_0 \dotplus \ldots \dotplus 2[m-1]_0 \dotplus (n-m+1)[m]_0,$$
*for* $n \geqslant m$.

Lastly, for $\mu \neq 0$, but $\nu = 0$, we have
$$a_{pq,p\,q+1} = \mu,$$
$$a_{pq,p+1\,q+1} = 1.$$

Each element in the rectangle may now be connected with the element immediately below it, or diagonally below it and to the right. For the maximum lengths of chains we must take each column as a chain. We obtain

III.
$$[n]_\mu \times [m]_0 = n[m]_0,$$
*for* $\mu \neq 0$.

It should be noted that square brackets [ ] denote a matrix; round brackets ( ) denote a scalar multiplier.

Our next step is to find the canonical form of the $m$th compound matrix $A^{[1^m]}$. We take first $A = [n]_\mu$.

The rows and columns of the compound matrix are defined by the sets of $m$ numbers $(p_1, p_2, \ldots, p_m)$ for which $1 \leqslant p_1 < \ldots < p_m \leqslant n$.

The canonical form of the compound matrix is found by the method of chains in the same manner as for direct products. Each set $(p_1, \ldots, p_m)$ is linked to another set in which any one of the $p$'s is increased by unity. The first set in the chain is clearly $(1, 2, \ldots, m)$, and the last $(n-m+1, n-m+2, \ldots, n)$. When the sets of the first chain are removed, the length of the greatest chain that can be made from the remaining sets gives the degree of the second largest submatrix, and so on. The greatest length of chain for all possible choices of earlier chains is taken in each case.

Each set $(p_1,\ldots, p_m)$ we now associate with the partition

$$(p_m-m, p_{m-1}-m+1,\ldots, p_1-1).$$

The sets are thus in one-one correspondence with all partitions $(\lambda_1,\ldots, \lambda_p)$ such that the number of parts is less than or equal to $m$ and the magnitude of each part less than or equal to $n-m$, including the partition $(0)$. The chains are obtained by adding unity to any of the parts.

The least partition is $(0)$ and the greatest $[(n-m)^m]$. Hence the length of the greatest chain is $m(n-m)+1$. This chain uses up one partition of every number from 0 to $m(n-m)$.

The next chain begins with a partition of the least number of which there are at least two partitions. If the beginning of the chain is known, the end of the chain may be determined, since for each partition $(\lambda_1,\ldots, \lambda_m)$ of $r$ into not more than $m$ parts each less than or equal to $n-m$, there is a corresponding partition $(n-m-\lambda_m,\ldots, n-m-\lambda_1)$ of $m(n-m)-r$. Hence, if $r_2$ is the least number of which there are two partitions into not more than $m$ parts each less than or equal to $n-m$, the maximum length of the second chain is $m(n-m)+1-2r_2$.

Similarly, the maximum length of the $i$th chain is $m(n-m)+1-2r_i$, where $r_i$ is the least number of which there are at least $i$ partitions into not more than $m$ parts each less than or equal to $n-m$.

This gives the maximum possible length for the chains. There is no guarantee that these lengths are attained, since, although there may remain partitions of two consecutive numbers, these may not link according to the rule, e.g. $(3)$ and $(2^2)$ do not link. We shall assume for the present that these lengths may be attained by a suitable choice of chains. This assumption will be justified later.

We thus arrive at

IV.

$$[n]_\mu^{[1^m]} = [m(n-m)+1]_{\mu^m} \dotplus c_1[m(n-m)-1]_{\mu^m} \dotplus \dotplus \ldots \dotplus c_r[m(n-m)-2r+1]_{\mu^m} \dotplus \ldots,$$

*where $c_r$ is the number of partitions of $r$ minus the number of partitions of $(r-1)$ into not more than $m$ parts each less than or equal to $n-m$, for $\mu \neq 0$.*

The case of a singular matrix, $\mu = 0$, is simpler. The only non-zero term in the $p_i$th row of $A$ is in the column $p_i+1$. Hence the criterion for the formation of chains is as follows: the set $(p_1,\ldots,\, p_m)$ is linked to $(p_1+1,\ldots,\, p_m+1)$. Hence any partition is linked to the partition in which each of the $m$ parts is increased by unity. It is convenient to deal with the conjugate partition. The conjugate partition $(\lambda_1,\ldots,\, \lambda_p)$ is linked to $(m,\, \lambda_1,\ldots,\, \lambda_p)$. The first partition of a chain must clearly have $\lambda_1 < m$, and the last have $n-m$ parts. The length of the chain is $n-m+1$ minus the number of parts in the first partition. Hence we have

V.
$$[n]_0^{[1^m]} = [n-m+1]_0 \dotplus c_1[n-m]_0 \dotplus c_2[n-m-1]_0 \dotplus \ldots \dotplus c_{n-m}[1]_0,$$
*where $c_i$ is the number of partitions into $i$ parts each less than or equal to $m-1$.*

To complete the solution of the problem for all compound matrices we need only

VI. $$[A \dotplus B]^{[1^m]} = [A]^{[1^m]} \dotplus [B]^{[1^m]} \dotplus \sum_{r=1}^{m-1} [A]^{[1^r]} \times [B]^{[1^{m-r}]}.$$

The proof is almost intuitive for any matrices. If $A$ is the matrix of transformation on $x_{1r},\, x_{2r},\ldots,\, x_{nr}$, and $B$ on $x_{n+1,r},\ldots,\, x_{n+n',r}$, for $1 \leqslant r \leqslant m$, then the matrix of transformation on the determinants $|x_{e_s t}|$ splits up into the direct sum of the matrices of transformation on determinants with assigned degrees in the first $n$ and the last $n'$ $x$'s respectively. The theorem follows readily.

EXAMPLE. Find the fourth compound matrix of $[4]_\mu \dotplus [3]_0 \dotplus [1]_\nu$.

From Theorems IV and V,
$$[4]_\mu^{[1^2]} = [5]_{\mu^2} \dotplus [1]_{\mu^2}, \quad [4]_\mu^{[1^3]} = [4]_{\mu^3}, \quad [4]_\mu^{[1^4]} = [1]_{\mu^4},$$
$$[3]_0^{[1^2]} = [2]_0 \dotplus [1]_0, \quad [3]_0^{[1^3]} = [1]_0.$$
Hence
$$\begin{aligned}\{[4]_\mu \dotplus [3]_0 \dotplus [1]_\nu\}^{[1^4]} \\ = [1]_{\mu^4} \dotplus [4]_{\mu^3}\{[3]_0 \dotplus [1]_\nu\} \dotplus \\ \dotplus \{[5]_{\mu^2} \dotplus [1]_{\mu^2}\} \times \{[2]_0 \dotplus [1]_0\} \dotplus \{[5]_{\mu^2} \dotplus [1]_{\mu^2}\} \times [3]_0 \times [1]_\nu \dotplus \\ \dotplus [4]_\mu \times [1]_0 \dotplus [4]_\mu \times \{[2]_0 \dotplus [1]_0\} \times [1] \dotplus [1]_0 \times [1]_\nu \\ = [1]_{\mu^4} \dotplus [4]_{\mu^3\nu} \dotplus 10[3]_0 \dotplus 10[2]_0 \dotplus 15[1]_0.\end{aligned}$$

**Induced and other invariant matrices**

The case of all other invariant matrices now presents no new difficulty. Since the spur of the direct product of two matrices is the product of the spurs, and the spur of a direct sum is the sum of the spurs, there is an isomorphism between invariant matrices and the corresponding $S$-functions. Thus, since

$$h_1^2 = h_2 + a_2,$$

the direct product of a matrix with itself is reducible by transformation into the direct sum of the second induced and the second compound matrices. Theorems I to VI, together with the rule for expressing any $S$-function as a determinant in the functions $a_r$, suffice for all cases.

For example, since

$$\{3, 1\} = \begin{array}{ccc} a_2, & a_3, & \\ 1, & a_1, & a_2 \\ & 1, & a_1 \end{array} = a_2 a_1^2 + a_4 - a_2^2 - a_1 a_3,$$

we have

$$[A]^{[3\,1]} = A^{[1^2]} \times A \times A \dot{+} A^{[1^4]} \dot{-} A^{[1^2]} \times A^{[1^2]} \dot{-} A \times A^{[1^3]}.$$

Putting $A = [5]$, substituting and evaluating by the above theorems, we obtain

$$[5]_\mu^{[3\,1]} = [15]_{\mu^4} \dot{+} [13]_{\mu^4} \dot{+} 2[11]_{\mu^4} \dot{+} 2[9]_{\mu^4} \dot{+} 3[7]_{\mu^4} \dot{+} 2[5]_{\mu^4} \dot{+} 2[3]_{\mu^4},$$

for $\mu \neq 0$.

The case of induced matrices could be solved in this manner, but it is simpler to start afresh from the method of chains.

The rows and columns of the induced matrix are defined by sets $(p_1, p_2, \ldots, p_m)$ in which repetitions are allowed. Chains are obtained in the same way by increasing one of the $p$'s by unity. Assuming that $p_1 \geqslant p_2 \geqslant \ldots \geqslant p_m$, the corresponding partition is

$$(p_1 - 1, p_2 - 1, \ldots, p_m - 1),$$

and the greatest magnitude for each part is $n-1$. Hence

VII. $$[n]_\mu^{[1^m]} = [n-m+1]_\mu^{[m]}.$$

This theorem also holds for $\mu = 0$. It is true independently of the assumption, made for Theorem IV, that the maximum lengths for the chains are attained. With the assumption we obtain

VIII.

$$[n]_\mu^{[m]} = [m(n-1)+1]_{\mu^m} \dot{+} c_1[m(n-1)-1]_{\mu^m} \dot{+} \\ \dot{+} \ldots \dot{+} c_r[m(n-1)-2r+1]_{\mu^m} \dot{+} \ldots,$$

*where* $c_r$ *is the number of partitions of* $r$ *minus the number of partitions of* $r-1$ *into not more than* $m$ *parts each less than or equal to* $n-1$, *for* $\mu \neq 0$.

IX. $[n]_0^{[m]} = [n]_0 \dotplus c_1[n-1]_0 \dotplus c_2[n-2]_0 \dotplus \dots \dotplus c_{n-1}[1]_0$, *where $c_r$ is the number of partitions into $r$ parts each less than or equal to $m-1$.*

### A generating function

The above results may be combined in the form of a generating function whose coefficients exhibit the canonical form of any invariant matrix.

The coefficient of $\rho^n x^m$ in

$$f(x) = 1/\{(1-x)(1-\rho x)\dots(1-\rho^p x)\}$$

is clearly the number of partitions of $n$ into not more than $m$ parts each less than or equal to $p$. Denote the product $(1-\rho)(1-\rho^2)\dots(1-\rho^r)$ by $[r]!$. The coefficient of $x^m$ in $f(x)$ is $[p+m]!/[p]![m]!$. Hence the number of partitions of $n$ into not more than $m$ parts each less than or equal to $p$ is equal to the coefficient of $\rho^n$ in $[p+m]!/[p]![m]!$.

### First correspondence rule

A polynomial in $\rho$,

$$\phi(\rho) = c_0 + c_1\rho + c_2\rho^2 + \dots + c_k\rho^k,$$

of reciprocal form, i.e. such that $c_i = c_{k-i}$, in which the positive integral coefficients $c_i$ are non-decreasing up to the middle term or terms, is associated with a matrix

$$c_0[k+1] \dotplus (c_1-c_0)[k-1] \dotplus (c_2-c_1)[k-3] \dotplus (c_3-c_2)[k-5] \dotplus \dots .$$

If $\phi(\rho)$ is the coefficient of a monomial expression in the roots of the characteristic equation of a matrix $A$, this monomial expression is in each case the leading diagonal term in the matrix.

X. *If $A = \sum [n_r]_{\mu_r}$, $\mu_r \neq 0$, let $f(x) = \prod f_r(x)$, where*

$$f_r(x) = 1/\{(1-\mu_r x)(1-\mu_r \rho x)\dots(1-\mu_r \rho^{n_r-1}x)\},$$

*and let $\{\lambda_1,\dots,\lambda_p\}$ be any S-function associated with the series*

$$f(x) = 1 + \sum h_r x^r.$$

*Then, if $\{\lambda_1,\dots,\lambda_p\} = \sum \phi_{\alpha\beta\gamma\dots}\mu_1^\alpha \mu_2^\beta\dots$, the classical canonical form of the matrix $A^{[\lambda_1,\dots,\lambda_p]}$ is the direct sum of the matrices associated with the coefficient $\phi_{\alpha\beta\gamma\dots}(\rho)$ according to the first correspondence rule.*

We first prove the theorem for the induced and compound matrices of $[n]_\mu$. The corresponding function is

$$f(x) = 1/\{(1-\mu x)(1-\mu\rho x)\dots(1-\mu\rho^{n-1}x)\}.$$

For this series

$$h_m = \frac{\mu^m[n+m-1]!}{[m]!\,[n-1]!}, \qquad a_m = \frac{\mu^m[n]!}{[m]!\,[n-m]!},$$

in which the coefficient of $\mu^m\rho_\mu$ is equal to the number of partitions of $p$ into not more than $m$ parts each less than or equal to $n-1$ in the case of $h_m$, and less than or equal to $n-m$ in the case of $a_m$. Theorems VIII and IV complete the proof for these cases.

To complete the proof in the general case it is only necessary to establish an isomorphism between the functions $\phi(\rho)$ and the corresponding matrices in regard to addition and multiplication.

The isomorphism is obvious in the case of addition provided that the two functions of $\rho$ that are added have the same mean degree. The mean degree of each symmetric function of the roots which goes to form $\phi_{\alpha\beta\gamma...}(\rho)$ is clearly $\frac{1}{2}\alpha(n_1+1)+\frac{1}{2}\beta(n_2+1)+\frac{1}{2}\gamma(n_3+1)+...$, and the condition is satisfied. It should be noted, however, that roots of the characteristic equation of $A$ which correspond to different elementary divisors must be kept symbolically distinct until the calculation is complete, or terms of different mean degree may be added and an incorrect result obtained.

For multiplication, it is clear that the product of $1+\rho+\rho^2+...+\rho^{n-1}$ by $1+\rho+...+\rho^{m-1}$ is in correspondence with the product $[n]\times[m]$. Hence the isomorphism and the theorem are established.

Singular submatrices require different treatment. We introduce a new series of symbols $\epsilon_1, \epsilon_2, \epsilon_3,...$, which are commutative and satisfy

$$\epsilon_i\epsilon_j = \epsilon_i \quad (i \leqslant j), \qquad \rho\epsilon_i = \epsilon_i, \qquad \mu_r\epsilon_i = \epsilon_i. \qquad (10.2;1)$$

**Second correspondence rule**

The polynomial $\psi(\epsilon) = \sum c_i\epsilon_i$, in which the positive integral coefficients $c_i$ satisfy $c_i \geqslant c_{i+1}$, is associated with the matrix

$$\sum (c_i-c_{i+1})[i]_0.$$

XI. *If* $A = \sum [n_r]_{\mu_r} \dotplus \sum [m_r]_0$, $\mu_r \neq 0$, *and* $\{\lambda_1,...,\lambda_p\}$ *is an S-function associated with* $f(x) = \prod f_r(x) \prod g_r(x)$, *where* $f_r(x)$ *is defined as in Theorem X, and* $g_r(x) = 1/\{(1-\epsilon_1 x)(1-\epsilon_2 x)...(1-\epsilon_{m_r} x)\}$, *then, if* $\{\lambda_1,...,\lambda_p\} = \sum \phi_{\alpha\beta\gamma...}(\rho)\mu_1^\alpha\mu_2^\beta...+\psi(\epsilon)$, *the classical canonical form of* $A^{[\lambda_1,...,\lambda_p]}$ *is the direct sum of the matrices associated with* $\phi_{\alpha\beta\gamma...}(\rho)$ *and* $\psi(\epsilon)$ *according to the first and second correspondence rules.*

The generating function which expresses the $S$-function as a coefficient is

$$\prod f(x_r)\Delta(x_1,\dots,x_n) = \sum \pm\{\lambda_1,\dots,\lambda_p\}x_1^{\lambda_1+n-1}x_2^{\lambda_2+n-2}\dots x_n^{\lambda_n}.$$

If we put $\rho = 1$, and $\epsilon_1 = \epsilon_2 = \dots = 0$, the $S$-function becomes the $S$-function of the roots of the characteristic equation.

To simplify calculation two rules are useful. For the $S$-function of a product $f(x) = f'(x)f''(x)$, denoting the $S$-functions respectively by $\{\lambda\}$, $\{\lambda\}'$, and $\{\lambda\}''$, we have

$$\{\lambda\} = \{\lambda\}'+\{\lambda\}''+\sum g_{\nu\mu\lambda}\{\nu\}'\{\mu\}'',$$

where $g_{\nu\mu\lambda}$ is the coefficient of $\{\lambda\}$ in the product $\{\nu\}\{\mu\}$.

The $S$-functions of $1/\{(1-x)(1-\rho x)\dots(1-\rho^{n-1}x)\}$ are given by

$$\{\lambda_1,\dots,\lambda_p\} = \frac{\prod(1-\rho^{\lambda_r-\lambda_s-r+s})}{\prod[\lambda_r+p-r]!}$$

times the product of the first $\lambda_i$ terms from each $i$th row of the set of numbers for which the $j$th term in the $i$th row is $\rho^{i-1}-\rho^{n+j-1}$.

As an example we evaluate $[[4]_\mu\dot{+}[2]_0]^{[3\,1]}$.

We have

$$\begin{aligned}[[4]_\mu\dot{+}[2]_0]^{[3\,1]} = [4]_\mu^{[3\,1]}\dot{+}[2]_0^{[3\,1]}\dot{+}\{[4]_\mu^{[3]}\dot{+}[4]_\mu^{[2\,1]}\}\times[2]_0\dot{+}\\ \dot{+}\{[4]_\mu^{[2]}\dot{+}[4]_\mu^{[1^2]}\}\times[2]_0^{[2]}\dot{+}[4]_\mu^{[2]}\times[2]_0^{[1^2]}\dot{+}\\ \dot{+}[4]_\mu\times\{[2]_0^{[2\,1]}\dot{+}[2]_0^{[3]}\}.\end{aligned}$$

$[4]_\mu^{[3\,1]}$ corresponds to the $S$-function $\{3\,1\}$ of

$$1/\{(1-\mu x)(1-\rho\mu x)(1-\rho^2\mu x)(1-\rho^3\mu x)\},$$

which is given by

$$\begin{aligned}\{3\,1\} &= \frac{\rho(1-\rho^3)(1-\rho^3)(1-\rho^4)(1-\rho^5)(1-\rho^6)}{(1-\rho)^2(1-\rho^2)(1-\rho^3)(1-\rho^4)}\\ &= \rho+2\rho^2+4\rho^3+5\rho^4+7p^5+7\rho^6+7\rho^7+5\rho^8+4\rho^9+2\rho^{10}+\rho^{11}.\end{aligned}$$

Hence $$[4]_\mu^{[3\,1]} = [11]_{\mu^4}\dot{+}[9]_{\mu^4}\dot{+}2[7]_{\mu^4}\dot{+}[5]_{\mu^4}\dot{+}2[3]_{\mu^4}.$$

All other submatrices are singular, and we may put $\rho = 1$, $\mu = 1$, so that the $S$-functions of $(1-x)^{-4}$ are taken.

We obtain

$$\begin{aligned}&[[4]_\mu\dot{+}[2]_0]^{[3\,1]}\dot{-}[4]_\mu^{[3\,1]}\\ &\quad= [2]_0^{[3\,1]}\dot{+}(20+20)[2]_0^{[1]}\dot{+}(10+6)[2]_0^{[2]}\dot{+}10[2]_0^{[1^2]}\dot{+}4[2]_0^{[2\,1]}\dot{+}4[2]_0^{[3]}.\end{aligned}$$

The corresponding function is

$$\begin{aligned}&\epsilon_1\epsilon_2(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2)+40(\epsilon_1+\epsilon_2)+16(\epsilon_1^2+\epsilon_1\epsilon_2+\epsilon_2^2)+\\ &\qquad+10\epsilon_1\epsilon_2+4\epsilon_1\epsilon_2(\epsilon_1+\epsilon_2)+4(\epsilon_1^3+\epsilon_1^2\epsilon_2+\epsilon_1\epsilon_2^2+\epsilon_2^3) = 105\epsilon_1+60\epsilon_2.\end{aligned}$$

Hence

$$[[4]_\mu \dotplus [2]_0]^{[31]} = [1\,1]_{\mu^4} \dotplus [9]_{\mu^4} \dotplus 2[7]_{\mu^4} \dotplus [5]_{\mu^4} \dotplus 2[3]_{\mu^4} \dotplus 60[2]_0 \dotplus 45[1]_0.$$

We now complete the proof of Theorems IV and VIII.

For any given $\mu \neq 0$, order the induced matrices $[n]_\mu^{[m]}$ in increasing order of $m+n$, and for fixed $m+n$, in increasing order of $n$. If all these matrices do not satisfy Theorem VIII, let $[n]_\mu^{[m]}$ be the first which does not.

For $r \leqslant m$, by Theorem VII, $[n]_\mu^{[1^r]} = [n-r+1]_\mu^{[r]}$, and, since $[n-r+1]_\mu^{[r]}$ satisfies Theorem VIII, $[n]^{[1^r]}$ will satisfy Theorem IV.

Now consider the equation

$$a_m - a_{m-1}h_1 + a_{m-2}h_2 - \ldots + (-1)^m h_m = 0, \qquad (10.2;\,2)$$

and the corresponding equation

$$[n]^{[1^m]} \dot{-} [n]^{[1^{m-1}]}[n] \dotplus [n]^{[1^{m-2}]}[n]^{[2]} \dot{-} \ldots \dotplus (-1)^m [n]^{[m]} = 0. \qquad (10.2;\,3)$$

The theoretical results given by Theorems IV and VIII must satisfy (10.2; 3), since the corresponding $S$-functions of

$$f(x) = 1/\{(1-x)(1-\rho x)\ldots(1-\rho^{n-1}x)\}$$

satisfy (10.2; 2). Again the correct results must satisfy (10.2; 3), since this equation is satisfied by all matrices. By our assumption, all matrices in (10.2; 3) except the last satisfy Theorems IV and VIII. Hence $[n]^{[m]}$ satisfies Theorem VIII, and we arrive at a contradiction.

Thus Theorems IV and VIII must hold in all cases.

## 10.3. Application to invariant theory (7)

It is the purpose of this section to show an application of invariant matrices to problems concerning the concomitants of polynomials.

It is beyond the scope of this book to discuss the foundations of invariant theory, and it must be assumed that the reader is aware of the nature and definitions of invariants, covariants, and other concomitants of polynomials, but beyond this no further assumption is necessary.

Let $f(X) = f(x_1, x_2, \ldots, x_r)$ be a homogeneous polynomial of degree $n$ in $r$ variables. Denote by $X$ a column-vector of which the elements are the $r$ variables $x_i$. Following the nomenclature for induced matrices, denote by $X^{[n]}$ the column-vector whose elements are the $\binom{n+r-1}{n}$ homogeneous products of the $x_i$'s of degree $n$.

Then $f(X)$ may be written in the form

$$f(X) = FX^{[n]},$$

where $F$ is a row-vector whose elements are the coefficients of $f(X)$.

If the $x_i$'s undergo a non-singular linear transformation, this may be expressed by

$$X = AX',$$

where $A$ is the matrix of transformation. Thus, by the definition of an induced matrix, we have

$$X^{[n]} = A^{[n]}X'^{[n]},$$

where $A^{[n]}$ is the $n$th induced matrix of $A$.

Hence
$$\begin{aligned} f(X) &= FX^{[n]} \\ &= FA^{[n]}X'^{[n]} \\ &= F'X'^{[n]}, \end{aligned}$$

where $F' = FA^{[n]}$.

Now consider the homogeneous products, of degree $s$, of the coefficients in $f(X)$. These are transformed according to the equation[†]

$$F'^{[s]} = F^{[s]}[A^{[n]}]^{[s]}.$$

Since the induced matrix of a given matrix is an invariant matrix, the induced matrix of an induced matrix must also be an invariant matrix of the original matrix, and hence is expressible as the direct sum of irreducible invariant matrices. Thus, the symbol of equality denoting identity of classical canonical form, let

$$[A^{[n]}]^{[s]} = \sum_{i=1}^{q} A^{[\lambda_i]},$$

$A^{[\lambda_i]}$ denoting the invariant matrix of $A$ which corresponds to the partition $(\lambda_i) \equiv (\lambda_{i1},\ldots,\lambda_{ip})$ of $ns$, and $\sum$ being used to denote direct sum. This direct sum may include, of course, repetitions and omissions.

Hence we can find $q$ row-vectors $P_1, P_2,\ldots, P_q$, whose elements are linear functions of the homogeneous products of degree $s$ of the coefficients of $f(X)$, such that these elements are linearly independent and that every homogeneous product of degree $s$ can be expressed linearly in terms of them; and when the $x_i$'s undergo a transformation

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[†] It should be noted that the induced matrix notation $A^{[n]}$, when applied to vectors, yields, for perfect consistency, a different usage for column-vectors and row-vectors. The column-vector has for elements the powers and products; but the row-vector has for elements powers and products multiplied respectively by the appropriate multinomial coefficients. This fits in well with the convention for invariants, where these multinomial coefficients are affixed to the coefficients of the forms. Hence there is a slight difference in the nature of $x^{[n]}$ and $F^{[s]}$.

$$X = AX',$$

these vectors undergo transformations

$$P_i' = P_i A^{[\lambda_i]}.$$

With two variables the only concomitants are invariants and covariants; with three variables, three new contragredient variables are introduced to express contravariants. With more than three variables, these are insufficient to express all concomitants. It is more convenient to dispense with contragredient variables, and express the concomitants in terms of sets of variables each cogredient with the original variables. Thus, if $X_1$, $X_2$,... are vectors cogredient with $X$, and $X_r = [x_{rs}]$, the contragredient variables may be replaced by the determinants

$$||x_{ts}||,$$

where $s$ runs from 1 to $n$, and $t$ from 1 to $n-1$. Generally, all concomitants may be expressed in terms of the $x_{ij}$'s.

With this convention for expressing a concomitant, it is clear that the types of concomitant correspond exactly to the Schur invariant matrices. For, let $\phi$ be any concomitant, and let $\phi = P\xi$, where $P$ is a row-vector of the coefficients and $\xi$ a column-vector with elements products of the $x_i$'s, or linear homogeneous functions of the $x_{ij}$. Under the transformation $X = AX'$, let $P$ and $\xi$ become $P'$ and $\xi'$, where

$$P' = PB \quad \text{and} \quad \xi = T(A)\xi'.$$

Then $T(A)$ is readily seen to be an invariant matrix of $A$. Also, since $P\xi$ is a concomitant (no generality being lost by taking $|A| = 1$), we have $P'\xi' = P\xi$, and so $B = T(A)$.

Hence the row-vector of coefficients is transformed by an invariant matrix of $A$. If $B = A^{[\lambda]}$, the concomitant is said to be of type $(\lambda)$.

I. *If*
$$[A^{[n]}]^{[s]} = \sum_{i=1}^{q} A^{[\lambda_i]},$$
*then the quantic of order $n$ possesses $q$ linearly independent concomitants of degree $s$, and these are of types $(\lambda_1)$, $(\lambda_2)$,..., $(\lambda_q)$ respectively.*

Put $\zeta = [x_{ts}]$, where $s$ and $t$ run from 1 to $r$. Then

$$\zeta = A\zeta',$$
$$\zeta^{[\lambda]} = A^{[\lambda]}\zeta'^{[\lambda]}.$$

For each of the $q$ vectors $P_i$ obtained above,

$$P'_i = P_i A^{[\lambda_i]}.$$

Hence $$P'_i \zeta'^{[\lambda_i]} = P_i A^{[\lambda_i]} \zeta'^{[\lambda_i]} = P_i \zeta^{[\lambda_i]},$$

and $P_i \zeta^{[\lambda_i]}$ is a concomitant of type $(\lambda_i)$.

If the number of parts in the partition $(\lambda_i)$ exceeds the number of variables, the concomitant vanishes identically.

### Concomitants of several polynomials

II. *If* $$\prod_{i=1}^{p} [A^{[n_i]}]^{[s_i]} = \sum_{i=1}^{q} A^{[\lambda_i]},$$

*then the system of $p$ quantics of orders $n_1, n_2,..., n_p$ respectively possesses $q$ concomitants of degrees $s_1, s_2,..., s_p$ in the respective quantics, and these are of type $(\lambda_1), (\lambda_2),..., (\lambda_q)$ respectively.*

The symbol $\prod$ is used for direct product. The proof is almost identical with that of Theorem I. If the $i$th quantic is

$$f_i(X) = F_i X_i^{[n_i]},$$

then the matrix of transformation on the homogeneous products of degree $s_i$ in the coefficients in each $f_i(X)$ is the direct product of each $s_i$th induced matrix of the matrix of transformation on the coefficients, and is therefore

$$\prod_{i=1}^{p} [A^{[n_i]}]^{[s_i]}.$$

### Digression on a new type of multiplication of $S$-functions

The problem of finding the number of concomitants of a given degree and type thus reduces to the essential problem of expressing the induced matrix of an induced matrix as the direct sum of irreducible invariant matrices.

A wider problem here suggests itself. Any invariant matrix of an invariant matrix must be an invariant matrix of the original matrix, and thus expressible as the direct sum of irreducible invariant matrices.

Thus $$[A^{[\lambda]}]^{[\mu]} = \sum k_{\lambda\mu\nu} A^{[\nu]}.$$

Hence we may define a new type of multiplication of $S$-functions

$$\{\lambda\} \otimes \{\mu\} = \sum k_{\lambda\mu\nu} \{\nu\}.$$

The multiplication is clearly associative, and distributive with respect to addition on the right only, i.e.

$$\{\lambda\} \otimes [\{\mu\} + \{\nu\}] = \{\lambda\} \otimes \{\mu\} + \{\lambda\} \otimes \{\nu\}.$$

For addition on the left we have

$$[\{\lambda\}+\{\mu\}]\otimes\{\nu\} = \sum g_{\nu_1\nu_2\nu}[\{\lambda\}\otimes\{\nu_1\}][\{\mu\}\otimes\{\nu_2\}],$$

where $g_{\nu_1\nu_2\nu}$ is the coefficient of $\{\nu\}$ in the (ordinary) product $\{\nu_1\}\{\nu_2\}$, the summation including the cases $\{\nu_1\} = \{0\} = 1$, $\{\nu_2\} = \{\nu\}$, and $\{\nu_1\} = \{\nu\}$, $\{\nu_2\} = \{0\} = 1$, and the multiplication between the brackets being the ordinary multiplication of $S$-functions.

### Generating functions for concomitants

The original problem of expressing the induced matrix of an induced matrix in terms of invariant matrices, and hence the problem of the concomitants, can be dealt with by the method of generating functions. Since the spurs of the invariant matrices are the $S$-functions of the roots of the characteristic equation of the original matrix, the problem may be expressed in the following form.

Take $r$ quantities $\alpha_1,\ldots, \alpha_r$. Form the $\binom{r+n-1}{n}$ quantities which are the homogeneous products of degree $n$. Then express the symmetric function $h_s$ of the second set of quantities as a sum of $S$-functions of $\alpha_1,\ldots, \alpha_r$.†

Thus, if $$[A^{[n]}]^{[s]} = \sum k_s^{(\lambda)} A^{[\lambda]},$$

the scalar coefficient $k_s^{(\lambda)}$ in the direct summation denoting the direct sum of $k_s^{(\lambda)}$ similar matrices $A^{[\lambda]}$, then

$$1/\prod(1-\alpha_1^{a_1}\alpha_2^{a_2}\alpha_3^{a_3}\ldots x) = 1+\sum_s\sum_\lambda k_s^{(\lambda)}\{\lambda\}x^s,$$

$\{\lambda\}$ being an $S$-function of $\alpha_1,\ldots, \alpha_r$, and the product being taken for all homogeneous products of degree $n$, $r^2$ being the order of the matrix $A$.

Multiplying by $\Delta(\alpha_1,\ldots,\alpha_r) = \prod(\alpha_i-\alpha_j)$ $(i<j)$, since

$$\{\lambda\}\Delta(\alpha_1,\ldots,\alpha_r) = \sum \pm\alpha_1^{\lambda_1+r-1}\alpha_2^{\lambda_2+r-2}\ldots\alpha_r^{\lambda_r},$$

we obtain

$$\Delta(\alpha_1,\ldots,\alpha_r)/\prod(1-\alpha_1^{a_1}\alpha_2^{a_2}\ldots\alpha_r^{a_r}x) = \sum_s\sum_\lambda \pm k_s^{(\lambda)}\alpha_1^{\lambda_1+r-1}\ldots\alpha_r^{\lambda_r}x^s.$$

Hence we have

III. *If*

$$\Delta(\alpha_1,\ldots,\alpha_r)/\prod(1-\alpha_1^{a_1}\ldots\alpha_r^{a_r}x) = \sum_s\sum_\lambda \pm k_s^{(\lambda)}\alpha_1^{\lambda_1+r-1}\ldots\alpha_r^{\lambda_r}x^s,$$

---

† It should be remembered that an $S$-function corresponding to a partition into more than $r$ parts vanishes identically.

*where the product on the left is taken with respect to all the homogeneous products of degree $n$, then the quantic of order $n$ in $r$ variables possesses exactly $k_s^{(\lambda)}$ concomitants of degree $s$ and type $(\lambda)$.*

This generating function was obtained by Young (8) by the use of the symbolic theory and quantitative substitutional analysis. The above method of obtaining the generating function is much simpler, and seems to shed light on its significance.

The generating function for a system of $q$ quantics is as follows.

IV. *If*

$$\Delta(\alpha_1,\dots,\alpha_r)/\prod(1-\alpha_1^{a_1}\alpha_2^{a_2}\dots\alpha_r^{a_r}x_i)$$
$$=\sum_s\sum_\lambda \pm k_{(s)}^{(\lambda)}\,\alpha_1^{\lambda_1+n-1}\alpha_2^{\lambda_2+n-2}\dots\alpha_r^{\lambda_r}x_1^{s_1}x_2^{s_2}\dots x_q^{s_q},$$

*where the product on the left is taken with respect to all the homogeneous products of degree $n_i$, for all $i$, then the system of $q$ quantics in $r$ variables, of orders $n_1$, $n_2$,..., $n_q$ respectively, possesses $k_{(s)}^{(\lambda)}$ concomitants of type $(\lambda)$ and degrees $s_1$,..., $s_q$ respectively in the quantics.*

**Binary concomitants**

The case of binary concomitants is exceptionally simple. The matrix of transformation has but two characteristic roots, $\alpha_1$ and $\alpha_2$. There is no loss of generality in taking $\alpha_1 = 1$; if we put $\alpha_2 = \rho$, the generating function becomes

$$\frac{1-\rho}{(1-x)(1-\rho x)(1-\rho^2x)\dots(1-\rho^{n-1}x)}. \tag{10.3; 1}$$

The coefficient of $x^s$ in this expression is well known to be

$$\frac{(1-\rho^n)(1-\rho^{n+1})\dots(1-\rho^{n+s-1})}{(1-\rho^2)(1-\rho^3)\dots(1-\rho^s)}. \tag{10.3; 2}$$

Hence the number of covariants of degree $s$ and order $q$ of the binary quantic of order $n$ is the coefficient of $\rho^{q+1}x^s$ in (10.3; 1), or the coefficient of $\rho^{q+1}$ in (10.3; 2).

We may obtain by this method a proof of the formula for the canonical form of an invariant matrix of a matrix with repeated characteristic roots, without the use of the method of chains.

Let $A$ be the matrix $\begin{bmatrix}\mu, 1\\ 0, \mu\end{bmatrix}$. Denote by $[n]_\mu$ the matrix of order $n^2$ with $\mu$ in each position in the leading diagonal, unity in each position in the diagonal next above, and zero elsewhere, so that $A = [2]_\mu$.

Clearly, confining our attention to the canonical form, we have

$$A^{[n]} = [n+1]_{\mu^n}.$$

Then, if $$[A^{[n]}]^{[s]} = \sum k_{pq} A^{[p,q]},$$

we have $$[n+1]^{[s]}_{\mu^n} = \sum k_{pq}[p-q+1]_{\mu^{ns}},$$

and $$[n+1]^{[s]}_{\mu} = \sum k_{pq}[p-q+1]_{\mu^s},$$

and the above generating function is applicable. The proof now follows the lines of the preceding section.

# XI

# GROUPS OF UNITARY MATRICES

**11.1.** To extend the orthogonal properties of the group characters to continuous matrix groups, the summation over the elements of the group must be replaced by an integration over the group manifold. The set of all matrices of order $n^2$ has a group manifold which is represented by a complete infinite $n^2$-dimensional space, and such integrals would necessarily diverge, so that the theory cannot be applicable.

For the theory to be applicable the group manifold must be represented as a closed finite space.

Consider first the case $n = 1$. The corresponding transformation is

$$y = kx.$$

If $k$ is real it must range from 0 to $\infty$, giving an infinite range of integration, and leading to the divergence of the integrals. But if we have

$$k = e^{i\theta},$$

with $\theta$ real, then $\theta$ may range between the finite limits $0 \leqslant \theta < 2\pi$.

The orthogonal properties of the group characters are in this case applicable, and lead to the theory of Fourier series.

For the general case of matrices of order $n^2$ it is necessary first that all characteristic roots should have modulus unity. Let $\lambda_1$ be a characteristic root of a matrix element $S$ of the group. Then $\lambda_1^n$ is a characteristic root of $S^n$.

If the group manifold is closed, then the sequence of points corresponding to

$$S,\ S^2,\ S^3,\ S^4,\ \ldots,\ S^n,\ \ldots$$

has a limit point corresponding to an element $S'$ of the group.

If $|\lambda_1| \neq 1$ then clearly $S'$ has either a zero or an infinite characteristic root. In the former case $S'$ has no inverse, so that in either case $S'$ cannot be regarded as a member of the group.

It is clear, then, that we require a group of matrices such that every characteristic root of every matrix element has modulus unity. Such a group is the *group of unitary matrices of order $n^2$*.

A unitary matrix $A = [a_{st}]$ and its elements satisfy the following relations (1.9; 1, 1.9; 2):

$$\bar{A}^t A = I, \qquad (11.1;1)$$

$$\left.\begin{aligned} \sum a_{rp}\,\bar{a}_{rp} &= 1, \\ \sum a_{rp}\,\bar{a}_{rq} &= 0 \quad (p \neq q), \end{aligned}\right\} \qquad (11.1;2)$$

the bar denoting the complex conjugate, and the suffix $t$ denoting that the transpose of the matrix is taken.

Putting $a_{rp} = a'_{rp} + ia''_{rp}$, the set of all complex matrices of order $n^2$ corresponds to the set of all the points in real infinite $2n^2$-dimensional space. The set of unitary matrices corresponds to the finite $n^2$-dimensional subspace of points which satisfy the $n^2$ equations (11.1; 2).

The group of unitary matrices of order $n^2$ has therefore a finite group manifold, and it will be seen that the orthogonal relations of the characters of finite groups may be extended to this group, and to various subgroups.

The general unitary matrix of order $n^2$ has $n^2$ elements $a_{st}$. A complex number $x$ and its conjugate complex $\bar{x}$ are *algebraically* independent, i.e. no polynomial relation exists between them. The $n^2$ elements $a_{st}$ and the $n^2$ conjugate complex quantities $\bar{a}_{st}$ satisfy the $n^2$ equations (11.1; 2). Hence there remain $n^2$ algebraically independent variable quantities in the general unitary matrix.

*The $n^2$ elements of the general unitary matrix are algebraically independent.*

It follows that every *algebraic* invariant of the group of unitary matrices of order $n^2$ is also an invariant of the group of all non-singular matrices of order $n^2$. Hence the following result holds.

*The complete set of independent irreducible matrix representations of the group of unitary matrices of order $n^2$ is given by the set of independent irreducible invariant matrices, and the simple characters of the group are the S-functions of the characteristic roots.*

Two subgroups of this group are especially important. The group of *real orthogonal matrices* is the subgroup which consists of the unitary matrices of which the elements are all real.

The orthogonal group has a subgroup which is the *rotation group*, and consists of those orthogonal matrices which have determinant $+1$.

We shall obtain the characters of these groups.

## 11.2. Fundamental formula for integration over the group manifold (1)

We have seen that the group manifold of the group of unitary matrices of order $n^2$ is an $n^2$-dimensional subspace of a $2n^2$-dimensional space. We now define the volume content of a subspace of the group manifold. Consider first infinitesimal unitary transformations.

In the neighbourhood of any point of the group manifold, the $n^2$-dimensional space of the manifold approximates to a linear space. The matrix of an infinitesimal unitary transformation may be expressed in the form $I+\Phi$, where

$$\Phi = [\phi_{st}]$$

is a skew-Hermitian matrix, so that

$$\phi_{pq} = -\bar{\phi}_{qp},$$

$$\phi_{pp} = \text{a pure imaginary.}$$

Let $$\phi_{pp} = i\phi''_{pp}$$

and $$\phi_{pq} = \phi'_{pq}+i\phi''_{pq} \quad (p \neq q).$$

If each element $\phi''_{pp}$, $\phi'_{pq}$, $\phi''_{pq}$ $(p < q)$ varies independently and continuously between the limits 0 and $\phi''_{pp}$, $\phi'_{pq}$, $\phi''_{pq}$ respectively, we define the volume content of that space of the group manifold which is so described by

$$|\Phi| = \prod \phi''_{pp} \prod_{p<q} \phi'_{pq}\phi''_{pq},$$

or the absolute value of this if it is negative.

We show first that the volume content is unaltered if $\Phi$ is transformed by a unitary matrix, i.e.

$$|S^{-1}\Phi S| = |\Phi|$$

if $S$ is a unitary matrix.

Let $|S^{-1}\Phi S| = k|\Phi|$. $k$ is a real positive number. If $k \neq 1$, we may clearly assume that $k > 1$, replacing $S$ by $S^{-1}$ if necessary.

Since the group manifold is approximately linear in the neighbourhood of the unit matrix, the equation

$$|S^{-1}\Phi S| = k|\Phi|$$

will be unaltered if $\Phi$ is replaced by $S^{-1}\Phi S$, so that

$$|S^{-2}\Phi S^2| = k^2|\Phi|,$$

and generally $$|S^{-n}\Phi S^n| = k^n|\Phi|.$$

Since the group manifold of unitary matrices forms a finite closed space, the set of points corresponding to the matrices $S$, $S^2$, $S^3$,..., $S^n$,... possess a limit point corresponding to the matrix $S_1$. Then clearly the volume content of $S_1^{-1}\Phi S_1$ is infinite if $k > 1$. This is impossible from our definition, so that we must have $k = 1$.

In a similar manner the volume content of a small subspace in the neighbourhood of the unit matrix of the group manifold of any continuous group of unitary matrices may be defined, and we shall have

$$|S^{-1}\Phi S| = |\Phi|$$

if $S$ is a matrix of the group.

Let $dS$ vary so that the matrix $S+dS$ of a unitary matrix group describes an element of volume of the group manifold in the neighbourhood of the matrix $S$. Correspondingly, $I+S^{-1}\,dS$ will describe an element of volume in the neighbourhood of the unit matrix.

We define the volume content of the space described by $S+dS$ to be

$$|dS| = |S^{-1}\,dS|.$$

It follows that if $T$ and $U$ are matrices of the group

$$|dS| = |T^{-1}\,dS| = |T^{-1}\,dSU| = |d(T^{-1}SU)|.$$

We have thus defined the volume content of any subspace of the group manifold. In integrals we omit the uprights and write simply $dS$. The volume content of the complete group manifold of a unitary group $G$ may thus be written

$$g = \int dS.$$

Now let $x(S)$ be any scalar function of the matrix element $S$. Then, if $T$ and $U$ are fixed matrix elements of $G$, as $S$ describes the complete group manifold, so also do $TS$ and $TSU$. Since

$$|dS| = |dTS| = |dTSU|,$$

we have

$$\begin{aligned}\int x(S)\,dS &= \int x(TS)\,dTS\\ &= \int x(TS)\,dS\\ &= \int x(TSU)\,dTSU\\ &= \int x(TSU)\,dS.\end{aligned}$$

We thus obtain the fundamental formula for integration over the group manifold:

$$\int x(S)\,dS = \int x(TS)\,dS = \int x(TSU)\,dS. \qquad (11.2;1)$$

This formula may clearly be extended to matrix functions of the matrix $S$, since each element of the matrix function is a scalar function.

### Reducibility (2)

Before proceeding to the extension of the orthogonal relations to unitary groups we prove certain general results concerning reducibility. We shall show that if a representation of a group by unitary matrices is *reducible*, it is also *completely reducible*. Further, we shall prove that every matrix representation of a unitary group, or of any group to which the methods of group integration are applicable, i.e. of a group with a finite closed group manifold, is equivalent to a unitary representation, so that for such groups reducibility implies complete reducibility.

Since the invariant matrices give the representations of the unitary group, it follows that for invariant matrices also, reducibility implies complete reducibility. If, however, the elements of the representation are not restricted to be polynomials in the elements of the original matrix as they are in the invariant matrices, then a representation of the full linear group may be reducible without being completely reducible, as an example will show.

Let $S_i$ be a matrix of order $n^2$, and let $\Delta_i$ be the modulus of its determinant. Then the representation

$$\begin{bmatrix} 1, & \log \Delta_i \\ 0, & 1 \end{bmatrix}$$

is reducible but not completely reducible.

*If a representation of a group by unitary matrices is reducible, it is also completely reducible.*

Let $M_i$ be a typical matrix of the representation. If the representation is reducible, then there is a matrix $B$ such that for all $i$

$$B^{-1}M_i B = \begin{matrix} M_i' & N_i \\ 0 & M_i'' \end{matrix}$$

where $M_i'$ and $M_i''$ are square matrices of orders, say, $p^2$ and $q^2$ respectively, with $p+q = n$. This implies that there is a $p$-dimensional subspace of the carrier space, namely, that which corresponds to the first $p$ columns of $B$ which is left invariant by the matrix $M_i$.

Clearly a unitary matrix $B'$ may be constructed such that the first $p$ columns belong to this subspace. Then, because of the relations

$$\sum_i b_{ij}' b_{ik}' = 0,$$

the $p$-dimensional subspace of the carrier space defines a unique *complementary* $n-p = q$-dimensional subspace, and the last $q$ columns of $B'$ belong to this subspace.

Clearly this $q$-dimensional subspace also is left invariant by the matrix $M_i$. It follows that $M_i$ is *completely* reduced by the matrix $B'$.

*Every matrix representation of a group to which the methods of group integration are applicable is equivalent to a unitary representation* (3).

Let $S_i$ be a typical element of the group and let $M_i$ be the corresponding matrix of a representation. Denote by $\tilde{M}_i$ the conjugate complex of the transposed matrix of $M_i$.

Let
$$V = \int \tilde{M}_i M_i \, dS_i$$

integrated over the complete group manifold. Then

$$\begin{aligned} \tilde{M}_j V M_j &= \int \tilde{M}_j \tilde{M}_i M_i M_j \, dS_i \\ &= \int \tilde{M}_j \tilde{M}_i M_i M_j \, d(S_i S_j) \\ &= V. \end{aligned}$$

Now $V$ is a Hermitian matrix of which the characteristic roots are essentially positive, for if $X$ is any vector

$$\tilde{X} V X = \int \tilde{X} \tilde{M}_i M_i X \, dS_i$$

is necessarily positive. If $A$ is a unitary matrix which transforms $V$ into diagonal form so that

$$A^{-1} V A = \operatorname{diag}(v_1, v_2, \ldots, v_n),$$

then, putting
$$A^{-1} T A = \operatorname{diag}(\sqrt{v_1}, \sqrt{v_2}, \ldots, \sqrt{v_n}),$$

we see that
$$V = \tilde{T} T.$$

Thus
$$\tilde{M}_j \tilde{T} T M_j = \tilde{T} T,$$
$$\tilde{T}^{-1} \tilde{M}_j \tilde{T} \, . \, T M_j T^{-1} = I,$$

and $T M_j T^{-1}$ is unitary, which proves the theorem.

This proof is applicable to finite groups, the integration being replaced by a summation.

**The orthogonal relations for the characters of unitary groups (4)**

Let $G$ be a group of unitary matrices $S$ of order $n^2$, and let

$$g = \int dS,$$

the summation being taken over the complete group manifold.

Let $H(S)$ be an irreducible matrix representation of $G$ of order $m^2$, and let $U = [U_{st}]$ be an arbitrary fixed matrix of the same order as $H(S)$.

Put
$$V = \int H(S^{-1})UH(S)\,dS.$$

Then, if $T$ is a fixed element of $G$,
$$\begin{aligned} H(T^{-1})VH(T) &= \int H(T^{-1}S^{-1})UH(ST)\,dS \\ &= \int H(T^{-1}S^{-1})UH(ST)\,dST \\ &= V. \end{aligned}$$

Thus $V$ commutes with every matrix $H(T)$. Since the representation is irreducible, it follows that
$$V = \alpha I,$$
where $\alpha$ is scalar.

Let $H_1(S)$ be any independent irreducible matrix representation of $G$ of order not greater than the order of $H(S)$, and suppose that it is bordered by rows and columns of zeros if the order is less.

Let
$$V_1 = \int H_1(S^{-1})UH(S)\,dS.$$

Then
$$\begin{aligned} H_1(T^{-1})V_1H(T) &= \int H_1(T^{-1}S^{-1})UH(ST)\,dST \\ &= V_1, \end{aligned}$$

and
$$V_1H(T) = H_1(T)V_1.$$

Since $H(T)$ and $H_1(T)$ are independent irreducible representations, this is impossible unless
$$V_1 = 0.$$

Thus we have
$$\int H(S^{-1})UH(S)\,dS = \alpha I, \tag{11.2;2}$$
$$\int H_1(S^{-1})UH(S)\,dS = 0. \tag{11.2;3}$$

Let
$$H(S) = [\sigma_{st}], \qquad H(S^{-1}) = [\tau_{st}],$$
$$H_1(S) = [\sigma'_{st}], \qquad H_1(S^{-1}) = [\tau'_{st}].$$

Picking out the element in the $a$th row, $b$th column of equation (11.2;2), we have
$$\sum_{p,q} \int \tau_{ap}\, u_{pq}\, \sigma_{qb}\, dS = \alpha\delta_{ab}, \tag{11.2;4}$$

where $\delta_{ab} = 1$ if $a = b$, and 0 otherwise. Put $a = b$ and sum for all $a$. Since

$$\sum \sigma_{qa}\tau_{ap} = \delta_{pq},$$

we have

$$\sum \int \delta_{pq} u_{pq}\, dS = \alpha m,$$

$$g \sum u_{pp} = \alpha m.$$

Equation (11.2; 4) may now be written

$$\sum_{p,q} \int \tau_{ap} u_{pq} \sigma_{qb}\, dS = \frac{g}{m} \sum u_{pp} \delta_{ab}.$$

From the coefficient of $u_{pq}$ we obtain

$$\frac{1}{g} \int \tau_{ap} \sigma_{qb}\, dS = \frac{1}{m} \delta_{ab} \delta_{pq}. \tag{11.2; 5}$$

Similarly, from (11.2; 3),

$$\int \tau'_{ap} \sigma_{qb}\, dS = 0. \tag{11.2; 6}$$

Hence, if $H(T) = [t_{st}]$,

$$\frac{1}{g} \int \chi(S)\chi(S^{-1}T)\, dS = \frac{1}{g} \int \sum \sigma_{ii} \tau_{pr} t_{rp}\, dS$$

$$= \frac{1}{m} \sum t_{ii} = \frac{1}{m} \chi(T).$$

In a similar manner from (11.2; 6) we obtain

$$\int \chi(S)\chi'(S^{-1}T)\, dS = 0.$$

Hence

$$\left.\begin{aligned} \frac{1}{g} \int \chi(S)\chi(S^{-1}T)\, dS &= \frac{1}{m}\chi(T), \\ \int \chi(S)\chi'(S^{-1}T)\, dS &= 0. \end{aligned}\right| \tag{11.2; 7}$$

From this we obtain the orthogonal relations by putting $T = I$:

$$\left.\begin{aligned} \frac{1}{g} \int \chi(S)\chi(S^{-1})\, dS &= 1, \\ \frac{1}{g} \int \chi(S)\chi'(S^{-1})\, dS &= 0. \end{aligned}\right\} \tag{11.2; 8}$$

## 11.3. Simplification of integration formulae for class functions (5)

Let $S$ be an element of a matrix group $G$, and let $x(S)$ be a scalar function of $S$. If, for every $T$ of $G$,

$$x(T^{-1}ST) = x(S),$$

then $x(S)$ is called a *class function*.

If $G$ is the group of unitary matrices of order $n^2$, we may choose a matrix $A$ of $G$ such that

$$A^{-1}SA = D \equiv \operatorname{diag}(e^{i\phi_1}, e^{i\phi_2},\ldots, e^{i\phi_n}).$$

Hence we may express $x(S)$ as

$$x(S) = x(D) = f(\phi_1, \phi_2,\ldots, \phi_n),$$

where $f$ is a periodic function with period $2\pi$ in each of the variables $\phi_1,\ldots, \phi_n$.

Now suppose that we make a small modification in $S$, so as to obtain the matrix $S+dS$. We must simultaneously make a small modification in $A$ to $A+dA$, so that

$$(A+dA)^{-1}(S+dS)(A+dA) = D+dD$$

is still in diagonal form.

The matrix $A+dA$ may be multiplied on the right by any diagonal (unitary) matrix, say $I+\delta$, without altering the diagonal matrix $D+dD$. If $\delta$ is small, neglecting second-order differences, $\delta$ is a pure imaginary diagonal matrix. The transforming matrix $A$ thus becomes

$$\begin{aligned} A' &= (A+dA)(I+\delta) \\ &= A+dA+A\delta \\ &= A(I+A^{-1}dA+\delta). \end{aligned}$$

Then we may choose $dA$ in one way and one way only so that in

$$\delta A = A^{-1}dA$$

the leading diagonal consists entirely of zeros.

Let
$$\delta A = [\epsilon_{st}] = [\epsilon'_{st}+i\epsilon''_{st}],$$
where
$$\epsilon'_{st} = -\epsilon'_{ts},$$
$$\epsilon''_{st} = \epsilon''_{ts}.$$

Then, as $S$ varies over a small $n^2$-dimensional volume of the group manifold, $\delta A$ describes a small $(n^2-n)$-dimensional volume. We define the volume content in this $(n^2-n)$-dimensional space, as $\epsilon'_{st}$, $\epsilon''_{st}$ vary independently between 0 and these limits, as

$$|\delta| = \prod_{i<j} \epsilon'_{ij}\epsilon''_{ij}$$

or the absolute value of this expression if negative.

Now we have
$$SA = AD,$$

and, for the modified matrix,

$$S(I+S^{-1}dS)A(I+\delta A) = A(I+\delta A)D(I+D^{-1}dD),$$

so that, neglecting second-order differences,

$$S(S^{-1}dS)A+SA\delta A = A\delta A\,D+AD.D^{-1}dD.$$

Multiplying by $\quad A^{-1}S^{-1} = D^{-1}A^{-1},$

$$A^{-1}(S^{-1}dS)A+\delta A = D^{-1}\delta A.D+D^{-1}dD,$$

$$A^{-1}(S^{-1}dS)A = D^{-1}\delta A\,D+D^{-1}dD-\delta A.$$

Now let $\quad D = \operatorname{diag}(\alpha_1, \alpha_2,\ldots, \alpha_n)..$

Then $\quad D^{-1}\delta A\,D-\delta A = \left[\left(\frac{\alpha_s}{\alpha_t}-1\right)\epsilon_{st}\right].$

Hence

$$|S^{-1}dS| = |A^{-1}(S^{-1}dS)A|$$

$$= \prod\left(\frac{\alpha_s}{\alpha_t}-1\right)|\delta A|\,d\alpha_1\,d\alpha_2\ldots d\alpha_n,$$

or the modulus of this expression.

Let $\alpha_r = e^{i\phi_r}$. Then the modulus of $d\alpha_r$ is $d\phi_r$. We have

$$\left(\frac{\alpha_s}{\alpha_t}-1\right)\left(\frac{\alpha_t}{\alpha_s}-1\right) = (e^{i\phi_s}-e^{i\phi_t})(e^{-i\phi_s}-e^{-i\phi_t}).$$

Put $x(S) = f(\phi_1,\ldots,\phi_n) \equiv f(\phi)$.

Then $\quad |S^{-1}dS| = \Delta(e^{i\phi_r})\Delta(e^{-i\phi_r})|\delta A|\,d\phi_1\,d\phi_2\ldots d\phi_n.$

We thus obtain

$$\frac{1}{h}\int f(\phi)\,dS = K\int\limits_0^{2\pi}\ldots\int\limits_0^{2\pi} f(\phi)\Delta(e^{i\phi_r})\Delta(e^{-i\phi_r})\,d\phi_1\ldots d\phi_n, \tag{11.3;1}$$

$K$ being a numerical factor.

This formula enables us to integrate a class function of the group of unitary matrices of order $n^2$, over the group manifold. $K$ may be evaluated by putting $f(\phi) = 1$.

We consider next the orthogonal group. First let $n$ be even, say $n = 2\nu$. Let $S$ be any member of the rotation group, i.e. an orthogonal matrix with determinant $+1$.

The canonical form of $S$ is

$$A^{-1}SA = D = \operatorname{diag}(e^{i\phi_1}, e^{-i\phi_1}, e^{i\phi_2}, e^{-i\phi_2},\ldots, e^{-i\phi_\nu}).$$

In general the transforming matrix $A$ is unitary, but not orthogonal.

If we restrict the transforming matrices to be orthogonal, the simplest form we can obtain is

$$A_1^{-1}SA_1 = D_1 = \mathrm{diag}(\Phi_1, \Phi_2,..., \Phi_\nu),$$

where
$$\Phi_r = \begin{bmatrix} \cos\phi_r, & \sin\phi_r \\ -\sin\phi_r, & \cos\phi_r \end{bmatrix}.$$

We put
$$d\Phi_r = \begin{bmatrix} 1, & d\phi_r \\ -d\phi_r, & 1 \end{bmatrix}.$$

Then
$$D_1^{-1}\,dD_1 = \mathrm{diag}(d\Phi_1, d\Phi_2,..., d\Phi_\nu).$$

The $n^2$ elements of a matrix are divided into three kinds. The first kind are the $n$ elements in the leading diagonal; the second kind are the $n$ elements either in row $(2r-1)$ and column $(2r)$, or in column $(2r-1)$ and row $(2r)$; the third kind are the $n^2-2n$ remaining elements.

The procedure is similar to that adopted for the unitary group. The elements of the first kind in $A_1^{-1}\,dA_1$ are identically zero, but $dA_1$ can be chosen uniquely so that the elements of the second kind are zero also.

Putting
$$E = \begin{bmatrix} 1, & -i \\ i, & 1 \end{bmatrix},$$

$$J = \mathrm{diag}(E^\nu),$$

then
$$D = J^{-1}D_1 J.$$

The transformation by $J$ transforms the $n^2-2n$ elements of the third kind in a matrix, in groups of four amongst themselves, and the volume content of the corresponding subspaces in a group manifold are unaltered. It follows readily that

$$|S^{-1}\,dS|$$
$$= \prod\left(\frac{e^{i\phi_r}}{e^{i\phi_s}}-1\right)\left(\frac{e^{i\phi_r}}{e^{-i\phi_s}}-1\right)\left(\frac{e^{-i\phi_r}}{e^{i\phi_s}}-1\right)\left(\frac{e^{-i\phi_r}}{e^{-\phi_s}}-1\right)|\delta A|\,d\phi_1\,d\phi_2\ldots d\phi_\nu.$$

Since

$$\left(\frac{e^{i\phi_r}}{e^{i\phi_s}}-1\right)\left(\frac{e^{i\phi_r}}{e^{-i\phi_s}}-1\right)\left(\frac{e^{-i\phi_r}}{e^{i\phi_s}}-1\right)\left(\frac{e^{-i\phi_r}}{e^{-i\phi_s}}-1\right)$$
$$= 16\sin^2\tfrac{1}{2}(\phi_r-\phi_s)\sin^2\tfrac{1}{2}(\phi_r+\phi_s)$$
$$= 4(\cos\phi_r-\cos\phi_s)^2,$$

we obtain

$$\frac{1}{h}\int f(\phi)\,dS = K\int_0^{2\pi}\dots\int_0^{2\pi} f(\phi)\Delta^2(\cos\phi_r)\,d\phi_1\dots d\phi_\nu, \quad (11.3;2)$$

where $K$ is a numerical constant.

For an orthogonal matrix $U$, of negative determinant, the characteristic roots are

$$1,\ -1,\ e^{i\psi_1},\ e^{-i\psi_1},\ \dots,\ e^{i\psi_{\nu-1}},\ e^{-i\psi_{\nu-1}}.$$

We obtain $$D_1 = \operatorname{diag}(1,-1,\Psi_1,\dots,\Psi_{\nu-1}),$$

where $$\Psi_r = \begin{bmatrix} \cos\psi_r, & \sin\psi_r \\ -\sin\psi_r, & \cos\psi_r \end{bmatrix},$$

and $$J = \operatorname{diag}(1^2, E^{\nu-1}).$$

The corresponding factor in this case is easily seen to be

$$\begin{aligned}\prod(1-e^{i\psi_r})(1-e^{-i\psi_r})(1+e^{i\psi_r})(1+e^{-i\psi_r})\prod 4(\cos\psi_r-\cos\psi_s)^2 \\ = \prod 16\sin^2\tfrac{1}{2}\psi_r\cos^2\tfrac{1}{2}\psi_r\prod 4(\cos\psi_r-\cos\psi_s)^2 \\ = \prod 4\sin^2\psi_r\prod 4(\cos\psi_r-\cos\psi_s)^2.\end{aligned}$$

We thus obtain

$$\frac{1}{h}\int f(\psi)\,dU = K\int_0^{2\pi}\dots\int_0^{2\pi} f(\psi)\prod(\sin^2\psi_r)\Delta^2(\cos\psi_r)\,d\psi_1\dots d\psi_{\nu-1}. \quad (11.3;3)$$

Lastly, for $n$ odd, say $n = 2\nu+1$, the characteristic roots of an orthogonal matrix of positive determinant are

$$1,\ e^{i\phi_1},\ e^{-i\phi_1},\ \dots,\ e^{i\phi_\nu},\ e^{-i\phi_\nu}.$$

The factor in this case is

$$\prod(1-e^{i\phi_r})(1-e^{-i\phi_r})\Delta^2(\cos\phi_r) = \prod 2(1-\cos\phi_r)\Delta^2(\cos\phi_r).$$

The matrix $U_1 = -I$ is an orthogonal matrix of negative determinant, and corresponding to every orthogonal matrix $S$ of positive determinant there is an orthogonal matrix of negative determinant $U = SU_1$, which may be defined by the same parameters $\phi_1,\dots,\phi_\nu$.

We thus obtain

$$\begin{aligned}\frac{1}{h}\int f(\phi)\,dS &= \frac{1}{h}\int f(\phi)\,dU \\ &= K\int_0^{2\pi}\dots\int_0^{2\pi} f(\phi)\prod(1-\cos\phi_r)\Delta^2(\cos\phi_r)\,d\phi_1\dots d\phi_\nu. \quad (11.3;4)\end{aligned}$$

If the characteristic roots of $U$ are

$$-1,\ e^{i\psi_1},\ e^{-i\psi_1},\ \dots,\ e^{i\psi_\nu},\ e^{-i\psi_\nu},$$

then clearly we may put

$$\psi_r = \pi + \phi_r. \tag{11.3; 5}$$

The corresponding formula to (11.3; 4) in terms of the parameters $\psi_r$ is

$$\frac{1}{h}\int f(\psi)\,dU = K\int_0^{2\pi}\dots\int_0^{2\pi} f(\psi)\prod(1+\cos\psi_r)\Delta^2(\cos\psi_r)\,d\psi_1\dots d\psi_\nu. \tag{11.3; 6}$$

## 11.4. Verification of the orthogonal properties of the characters of the unitary group

Since the irreducible matrix representations of the unitary group are the sets of invariant matrices, the simple characters of this group are the $S$-functions of the characteristic roots.

Let the characteristic roots of a unitary matrix $S$ be

$$e^{i\phi_1},\ e^{i\phi_2},\ \dots,\ e^{i\phi_n}.$$

The orthogonal relations for the characters of the unitary group may be expressed

$$\int_0^{2\pi}\dots\int_0^{2\pi}\{\lambda\}\overline{\{\lambda\}}\Delta(e^{i\phi_r})\Delta(e^{-i\phi_r})\,d\phi = K^{-1},$$

$$\int_0^{2\pi}\dots\int_0^{2\pi}\{\lambda\}\overline{\{\mu\}}\Delta(e^{i\phi_r})\Delta(e^{-i\phi_r})\,d\phi = 0,\quad \{\lambda\}\neq\{\mu\},$$

$\overline{\{\lambda\}}$ denoting the complex conjugate of $\{\lambda\}$.

Putting $\alpha_r = e^{i\phi_r}$, we have, from

$$\{\lambda\}\Delta(\alpha_r) = |\alpha_s^{\lambda_t+n-t}| = \sum\pm\prod\alpha_r^{\lambda_{p_r}+n-p_r},$$

and since $\bar{\alpha}_r = \alpha_r^{-1}$,

$$\overline{\{\mu\}}\Delta(\alpha_r^{-1}) = |\alpha_s^{-\mu_t-n+t}| = \sum\pm\prod\alpha_r^{-p_{q_r}-n+q_r},$$

the summations being taken with respect to all permutations $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)$ of $(1, 2,\dots, n)$, taking a minus sign for a negative permutation.

Hence

$$\int^{2\pi}\dots\int^{2\pi}\{\lambda\}\overline{\{\mu\}}\Delta(\alpha_r)\Delta(\alpha_r^{-1})\,d\phi = \sum\pm\prod\int^{2\pi}\alpha_r^{\lambda_{p_r}+n-p_r-\mu_{q_r}-n+q_r}\,d\phi_r.$$

The result is clearly zero unless every index is zero, i.e. unless

$$(\lambda) = (\mu)$$

and $$p_r = q_r \quad (r = 1,\dots, n).$$

We thus have

$$\int_0^{2\pi} \dots \int_0^{2\pi} \{\lambda\}\overline{\{\mu\}}\Delta(e^{i\phi_r})\Delta(e^{-i\phi_r})\,d\phi = 0, \quad \{\lambda\} \neq \{\mu\},$$

$$\int_0^{2\pi} \dots \int_0^{2\pi} \{\lambda\}\overline{\{\lambda\}}\Delta(e^{i\phi_r})\Delta(e^{-i\phi_r})\,d\phi = n!\,(2\pi)^n$$

The last result is independent of the particular $S$-function chosen, and we may put $\{\lambda\} = \{0\} = 1$. We thus obtain

$$\left.\begin{aligned} \int \{\lambda\}\overline{\{\mu\}}\,dS &= 0, \\ \frac{1}{h}\int \{\lambda\}\overline{\{\lambda\}}\,dS &= 1. \end{aligned}\right\} \qquad (11.4;1)$$

## 11.5. Orthogonal matrices and the rotation groups (6)

We denote by $D'$ the group of orthogonal matrices of order $n^2$, by $D$ the subgroup consisting of the matrices with positive determinant, and by $D_1$ the set of matrices with negative determinant, so that

$$D' = D + D_1.$$

We use the letter $S$ to denote a variable element of $D$, and the letter $U$ to denote a variable element of $D_1$. The letter $T$ denotes a variable element of $D'$ without reference to the sign of its determinant. Thus for integration over the complete group manifold we may write

$$\int \phi(T)\,dT = \int \phi(S)\,dS + \int \phi(U)\,dU.$$

If $U_1$ is a fixed element of $D_1$, for every element $S$ of $D$ there is an element $U = U_1 S$ of $D_1$, and conversely. Hence, putting

$$h = \int dS,$$

we have $$\int dU = \int dS = h,$$

and $$\int dT = 2h.$$

An orthogonal matrix cannot, in general, be transformed into diagonal form by another orthogonal matrix, since the characteristic roots are, in general, complex numbers. It may, however, be transformed by an orthogonal matrix into one of the four forms:

$$\operatorname{diag}(\Phi_1,\ldots,\Phi_\nu),\qquad \operatorname{diag}(1,-1,\Psi_1,\ldots,\Psi_{\nu-1}),$$
$$\operatorname{diag}(1,\Phi_1,\ldots,\Phi_\nu),\qquad \operatorname{diag}(-1,\Psi_1,\ldots,\Psi_\nu),$$

where

$$\Phi_r = \begin{bmatrix} \cos\phi_r, & \sin\phi_r \\ -\sin\phi_r, & \cos\phi_r \end{bmatrix},\qquad \Psi_r = \begin{bmatrix} \cos\psi_r, & \sin\psi_r \\ -\sin\psi_r, & \cos\psi_r \end{bmatrix}.$$

It is clear, then, that two orthogonal matrices with the same characteristic equation may be transformed into one another by orthogonal matrices. A class function of the orthogonal group may thus be expressed as a function of the characteristic roots.

Four cases arise. If $n$ is even, say $n = 2\nu$, the characteristic roots of a matrix $S$ of $D$ are of the form

$$e^{i\phi_1},\ e^{-i\phi_1},\ \ldots,\ e^{i\phi_\nu},\ e^{-i\phi_\nu}.$$

A class function may thus be expressed, for the elements of $D$, in the form

$$f(\phi) \equiv f(\phi_1,\phi_2,\ldots,\phi_\nu).$$

Since the characteristic roots $e^{i\phi_r}$, $e^{-i\phi_r}$ are interchangeable, $f(\phi)$ must be an even function of each parameter $\phi_r$. It is also periodic in each parameter, with period $2\pi$.

The characteristic roots of a matrix $U$ of $D_1$ are of the form

$$1,\ -1,\ e^{i\psi_1},\ e^{-i\psi_1},\ \ldots,\ e^{i\psi_{\nu-1}},\ e^{-i\psi_{\nu-1}}.$$

A class function may be expressed, for the elements of $D_1$, in the form

$$g(\psi) \equiv g(\psi_1,\psi_2,\ldots,\psi_{\nu-1}).$$

For odd $n$, say $n = 2\nu+1$, we may take $U_1 = -I$, and for each matrix $S$ of $D$ there is a matrix $U = U_1 S$ of $D_1$. The same parameters may be used for the two matrices $U$ and $S$. The characteristic roots of $S$ are of the form

$$1,\ e^{i\phi_1},\ e^{-i\phi_1},\ \ldots,\ e^{i\phi_\nu},\ e^{-i\phi_\nu},$$

and a class function may be expressed, for the elements of $D$, in the form

$$f(\phi) \equiv f(\phi_1,\ldots,\phi_\nu),$$

and for the elements of $D_1$, in the form

$$g(\phi) \equiv g(\phi_1,\ldots,\phi_\nu).$$

The class functions in each case are even functions of each parameter with period $2\pi$.

Two elements of $D$, $S_1$ and $S_2$, which are equivalent in $D'$ may not be equivalent in $D$, for the matrices which transform $S_1$ into $S_2$ may all belong to $D_1$. Thus for $n = 2$, the matrices

$$\begin{bmatrix} \cos\theta, & \sin\theta \\ -\sin\theta, & \cos\theta \end{bmatrix}, \qquad \begin{bmatrix} \cos\theta, & -\sin\theta \\ \sin\theta, & \cos\theta \end{bmatrix}$$

may be transformed into one another by any orthogonal matrix

$$\begin{bmatrix} \cos\phi, & \sin\phi \\ \sin\phi, & -\cos\phi \end{bmatrix}$$

of negative determinant, but not by an orthogonal matrix of positive determinant.

For $n = 2\nu+1$, two matrices $S_1$ and $S_2$ which are equivalent in $D'$, are equivalent in $D$ also, for if

$$S_2 = U^{-1}S_1 U,$$

then also

$$S_2 = (UU_1)^{-1}S_1(UU_1),$$

and $UU_1$ belongs to $D$.

For $n = 2\nu$, however, a class of $D'$ separates, in general, into two *conjugate* classes of $D$. We may still denote a class function by

$$f(\phi) \equiv f(\phi_1,\dots,\phi_\nu),$$

this denoting a function of the class of matrices equivalent to the matrix

$$\mathrm{diag}(\Phi_1, \Phi_2,\dots,\Phi_\nu)$$

where

$$\Phi_r = \begin{bmatrix} \cos\phi_r, & \sin\phi_r \\ -\sin\phi_r, & \cos\phi_r \end{bmatrix}.$$

The function $f(\phi)$ is not necessarily an even function of each parameter, but if the sign of an even number of the parameters is changed, then $f(\phi)$ is unaltered.

The value of $f(\phi)$ for the conjugate class is obtained by changing the sign of an odd number of the parameters.

It follows that if $\nu$ is odd, so that $n$ is of the form $4r+2$, $S$ and $S^{-1}$ belong to conjugate classes; but for even $\nu$, so that $n$ is of the form $4r$, $S$ and $S^{-1}$ belong to the same class.

A class function $f(\phi)$ which is an even function of each parameter is called an *even class function*. It takes the same value for two conjugate classes, and is clearly a class function of $D'$ also.

## 11.6. Relations between the characters of $D$ and $D'$

Let $\chi(T)$ be a simple character of $D'$. Since $T$ and $T^{-1}$ have the same characteristic roots, they are equivalent in $D'$, and

$$\chi(T) = \chi(T^{-1}) = \overline{\chi(T)},$$

the bar denoting the complex conjugate. It follows that the characters of $D'$ are all real.

Since $D$ is a subgroup of $D'$, it follows that $\chi(S)$ is a character, simple or compound, of $D$. Now

$$\int \chi(T)^2\,dT = 2h = \int \chi(S)^2\,dS + \int \chi(U)^2\,dU.$$

Hence, either $\chi(S)$ is a simple character of $D$, and

$$\int \chi(S)^2\,dS = \int \chi(U)^2\,dU = h,$$

or else $\chi(S)$ is the sum of two distinct simple characters of $D$,

$$\chi(S) = \chi_1(S)+\tilde{\chi}_1(S),$$

so that

$$\int \chi(S)^2\,dS = 2h,$$

and

$$\chi(U) = 0,$$

for every $U$.

In the latter case $\chi(T)$ is called a *double character* of $D'$.

Let $\chi(T)$ be a double character of $D'$, and let

$$\chi(S) = \chi_1(S)+\tilde{\chi}_1(S).$$

If $n = 2\nu+1$, two elements of $D$ which are equivalent in $D'$ are equivalent in $D$ also. The classes of $D$ coincide with the classes of matrices of positive determinant in $D'$. $\chi_1(S)$ is thus a class function of $D'$, and is expressible in terms of the characters of $D'$. This is clearly inconsistent with the orthogonal relations.

*If $n = 2\nu+1$, there are no double characters of $D'$.*

Similarly, for $n = 2\nu$, $\chi_1(S)$ cannot take the same value for every pair of conjugate classes of $D$. If $\tilde{S}$ is an element of the class conjugate to the class containing $S$, $\chi_1(\tilde{S})$ is a simple character of $D$, not equal to $\chi_1(S)$. Since

$$\int \chi(S)\chi_1(\tilde{S})\,dS = \int \chi(\tilde{S})\chi_1(\tilde{S})\,dS = h,$$

it follows that

$$\chi_1(\tilde{S}) = \tilde{\chi}_1(S).$$

If $\nu$ is odd, $S$ and $S^{-1}$ are members of conjugate classes of $D$, and we may take $\tilde{S} = S^{-1}$. Hence

$$\tilde{\chi}_1(S) = \chi_1(S^{-1}) = \bar{\chi}_1(S),$$

the conjugate complex of $\chi_1(S)$.

*If $n$ is of the form $4r+2$, every double character of $D'$ is the sum of two conjugate complex characters of $D$.*

On the other hand, if $\nu$ is even, $S$ and $S^{-1}$ are equivalent in $D$, and $\chi_1(S^{-1}) = \chi_1(S) = \bar{\chi}_1(S)$.

*If $n$ is of the form $4r$, all characters of $D$ are real.*

Denote by $\chi^{(0)}(T)$ the character of $D'$ which is unity for every element,

$$\chi^{(0)}(T) = 1.$$

There is also a character of $D'$, which we denote by $\chi^{(0)*}(T)$, which is equal to the determinant of the matrix $T$,

$$\chi^{(0)*}(S) = +1, \qquad \chi^{(0)*}(U) = -1.$$

Then, if $\chi(T)$ is any simple character of $D'$, so also is $\chi(T)\chi^{(0)*}(T)$. If $\chi(T)$ is a double character

$$\chi(T)\chi^{(0)*}(T) = \chi(T),$$

since

$$\chi(U) = 0.$$

If $\chi(T)$ is not a double character

$$\chi(T)\chi^{(0)*}(T) = \chi^*(T),$$

a distinct simple character.

$\chi(T)$ and $\chi^*(T)$ are called *associated* characters of $D'$.

The characters of $D$ and $D'$ thus fall into two categories.

There are pairs of associated characters of $D'$ which correspond to the same simple character of $D$.

There are double characters of $D'$, satisfying $\chi(U) = 0$, which separate into two conjugate characters of $D$. If $n$ is of the form $4r+2$, these are conjugate complex characters. If $n$ is of the form $4r$, they are distinct real characters. If $n$ is odd there are no double characters of $D'$.

## 11.7. Integration formulae connected with $D$ and $D'$

Let $f(x) = 0$ be the characteristic equation of an element $T$ of $D'$, and let

$$f_i = f(x_i), \qquad f'_i = f(y_i).$$

We proceed to evaluate the integrals

$$2hJ = \int \frac{1}{f_1 f_2 \cdots f_\nu f'_1 \cdots f'_\nu}\, dT,$$

$$hJ_1 = \int \frac{1}{f_1 f_2 \cdots f_\nu f'_1 \cdots f'_\nu}\, dS.$$

It is convenient to put

$$2hJ_2 = \frac{1}{f_1 f_2 \cdots f_\nu f'_1 \cdots f'_\nu}\, dU.$$

We require six lemmas, which follow.

**Lemma I.**
$$\frac{1}{1-x_s y_t} = \frac{\Delta(x)\Delta(y)}{\prod(1-x_i y_j)},$$
*where* $\Delta(x) = \prod_{i<j}(x_i - x_j)$, $\Delta(y) = \prod_{i<j}(y_i - y_j)$.

This was proved in § 5.2.

**Lemma II.**

$$\left|\lambda + \frac{\mu}{1-x_s y_t}\right| = \frac{\mu^{n-1}\Delta(x)\Delta(y)}{\prod(1-x_i y_j)}\{\mu + \lambda[1-\prod(x_i y_i)]\}.$$

By subtracting the first row from each subsequent row the determinant is seen to be linear in $\lambda$. It is also homogeneous of the $n$th degree in $\lambda$ and $\mu$ together. The lemma reduces to Lemma I for the case $\lambda = 0$. Also for $\lambda = -1$, $\mu = 1$ we obtain

$$\left.-1 + \frac{1}{1-x_s y_t}\right| = \left|\frac{x_s y_t}{1-x_s y_t}\right| = \prod(x_i y_i)\;\overline{1-x_s y_t}$$

This case is also proved by Lemma I. The truth in the general case follows.

**Lemma III.**

$$\left|\overline{1-2x_s\cos\phi_t + x_s^2}\right. = \frac{2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\phi)\Delta(x)L(x)}{\prod(1-2x_i\cos\phi_j + x_i^2)},$$

*where* $L(x) = \prod(1-x_i x_j)$.

We have

$$\overline{1-2x_s\cos\phi_t + x_s^2} \quad \overline{\prod(1+x_i^2)}\; 1-\frac{2x_s}{1+x_s^2}\cos\phi_t$$

$$\frac{1}{\prod(1+x_i^2)} \cdot \frac{\Delta(\cos\phi)\prod\limits_{i<j}\left(\frac{2x_i}{1+x_i^2} - \frac{2x_j}{1+x_j^2}\right)}{\prod\left(1-\frac{2x_i}{1+x_i^2}\cos\phi_j\right)}$$

$$= \frac{1}{\prod(1+x_i^2)} \cdot \frac{\Delta(\cos\phi)2^{\frac{1}{2}(\nu^2-\nu)}\prod(1+x_i^2)^{-\nu+1}\prod(x_i - x_j)(1-x_i x_j)}{\prod(1+x_i^2)^{-\nu}\prod(1-2x_i\cos\phi_j + x_i^2)}$$

$$= \frac{2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\phi)\Delta(x)L(x)}{\prod(1-2x_i\cos\phi_j + x_i^2)}.$$

LEMMA IV.

$$\int_0^{2\pi} \frac{d\phi}{(1-2x\cos\phi+x^2)(1-2y\cos\phi+y^2)} = \frac{2\pi(1+xy)}{(1-x^2)(1-y^2)(1-xy)},$$

if $|x|$, $|y| < 1$.

Taking $\Gamma$ as the contour $|z| = 1$, the required integral is equal to

$$\int_\Gamma \frac{1}{(1-xz)(1-xz^{-1})(1-yz)(1-yz^{-1})} \frac{dz}{iz},$$

as will be seen by putting $z = re^{i\phi}$. This integral is equal to

$$\int_\Gamma \frac{-iz\,dz}{(1-xz)(z-x)(1-yz)(z-y)}$$

$$= 2\pi i \text{ times sum of residues at } z = x,\ z = y,$$

$$= +2\pi\left[\frac{x}{(1-x^2)(1-xy)(x-y)} + \frac{y}{(1-y^2)(1-xy)(y-x)}\right]$$

$$= \frac{2\pi(x-xy^2-y+x^2y)}{(1-x^2)(1-y^2)(1-xy)(x-y)}$$

$$= \frac{2\pi(1+xy)}{(1-x^2)(1-y^2)(1-xy)}.$$

LEMMA V.

$$\int_0^{2\pi} \frac{(1-\cos^2\phi)\,d\phi}{(1-2x\cos\phi+x^2)(1-2y\cos\phi+y^2)(1-2u\cos\phi+u^2)(1-2v\cos\phi+v^2)}$$
$$= \frac{\pi(1-xyuv)}{(1-xy)(1-xu)(1-xv)(1-yu)(1-yv)(1-uv)}$$

for $|x|$, $|y|$, $|u|$, $|v| < 1$.

LEMMA VI.

$$\int_0^{2\pi} \frac{(1-\cos\phi)\,d\phi}{(1-2x\cos\phi+x^2)(1-2y\cos\phi+y^2)} = \frac{\pi}{(1+x)(1+y)(1-xy)}$$

for $|x|$, $|y| < 1$.

These two lemmas are easily proved by contour integration in a manner similar to Lemma IV.

For $n = 2\nu$, we have, for an element of $D$

$$f_i = \prod_j (1-2x_i\cos\phi_j+x_i^2),$$
$$f_i' = \prod_j (1-2y_i\cos\phi_j+y_i^2).$$

Hence

$$\Delta(x)L(x)\Delta(y)L(y)J_1 = \int \frac{\Delta(x)L(x)}{f_1\cdots f_\nu}\,\frac{\Delta(y)L(y)}{f'_1\cdots f'_\nu}\,dS$$

$$= K\int_0^{2\pi}\cdots\int_0^{2\pi} \frac{\Delta(\cos\phi)\Delta(x)L(x)}{f_1\cdots f_\nu}\,\frac{\Delta(\cos\phi)\Delta(y)L(y)}{f'_1\cdots f'_\nu}\,d\phi$$

$$\frac{K}{2^{\nu^2-\nu}}\int^{2\pi}\cdots\int^{2\pi} \frac{}{1-2x_s\cos\phi_t+x_s^2}\quad\frac{}{1-2y_s\cos\phi_t+y_s^2}\,d\phi$$

$$= \frac{K}{2^{\nu^2-\nu}}\sum\pm\prod\int^{2\pi}\frac{d\phi_i}{(1-2x_{\lambda_i}\cos\phi_i+x_{\lambda_i}^2)(1-2y_{\mu_i}\cos\phi_i+y_{\mu_i}^2)},$$

summed for every pair of permutations $(\lambda_1,\ldots,\lambda_\nu)$, $(\mu_1,\ldots,\mu_\nu)$ of $(1,2,\ldots,\nu)$,

$$= \frac{(2\pi)^\nu K}{2^{\nu^2-\nu}}\sum\pm\prod\frac{1+x_{\lambda_i}y_{\mu_i}}{1-x_{\lambda_i}y_{\mu_i}}\,\frac{1}{(1-x_{\lambda_i}^2)(1-y_{\mu_i}^2)}$$

$$= \frac{(2\pi)^\nu K}{2^{\nu^2-\nu}\prod[(1-x_i^2)(1-}\qquad\frac{1+x_sy_t|}{1-x_sy_t}$$

$$= \frac{(2\pi)^\nu K}{2^{\nu^2-\nu}\prod[(1-x_i^2)(1-y_i^2)]}\Big|{-1}+\frac{}{1-x_sy_t}$$

$$= \frac{K'\Delta(x)\Delta(y)[1+\prod(x_iy_i)]}{\prod[(1-x_i^2)(1-y_i^2)]\prod(1-x_iy_j)}.$$

Removing the factors $\Delta(x)$, $\Delta(y)$ from both sides of the equation and putting $x_i = y_i = 0$ for each $i$, we see that $K' = 1$. We thus obtain

$$L(x)L(y)J_1 = \frac{L(x)L(y)}{h}\int\frac{1}{f_1\cdots f_\nu f'_1\cdots f'_\nu}\,dS$$
$$= \frac{1+\prod(x_iy_i)}{\prod[(1-x_i^2)(1-y_i^2)]\prod(1-x_iy_j)}. \tag{11.7;1}$$

Next, for an element of $D_1$, we have

$$f_i = (1-x_i^2)\prod_j(1-2x_i\cos\psi_j+x_i^2),$$

$$f'_i = (1-y_i^2)\prod_j(1-2y_i\cos\psi_j+y_i^2).$$

We denote by $\Delta'(x)$ the product $\prod_{i<j}(x_i-x_j)$, in which the variable

$x_\nu$ is omitted. $\Delta'(y)$, $L'(x)$, $L'(y)$ are defined similarly. We thus obtain

$$\Delta'(x)L'(x)\Delta'(y)L'(y)I_2 = \frac{1}{h}\int \frac{1}{f_\nu f'_\nu}\,\frac{\Delta'(x)L'(x)}{f_1\cdots f_{\nu-1}}\,\frac{\Delta'(y)L'(y)}{f'_1\cdots f'_{\nu-1}}\,dU$$

$$= K\int\limits^{2\pi}\cdots\int\limits^{2\pi}\frac{\prod(1-\cos^2\psi_r)}{\prod[(1-x_i^2)(1-y_i)^2]}\,\frac{\Delta'(x)L'(x)\Delta(\cos\psi)}{\prod(1-2x_i\cos\psi_j+x_i^2)}\times$$

$$\times\frac{\Delta'(y)L'(y)\Delta(\cos\psi)}{\prod(1-2y_i\cos\psi_j+y_i^2)}\,d\psi$$

$$= \frac{K}{2^{\nu^2-\nu}}\int\limits^{2\pi}\cdots\int\limits^{2\pi}\frac{}{\prod[(1-x_i^2)(1-}\quad\times$$

$$\times\frac{\prod(1-\cos^2\psi_r)}{\prod[(1-2x_\nu\cos\psi_j+x_\nu^2)(1-2y_\nu\cos\psi_j+y_\nu^2)]}\times$$

$$\times\;\frac{}{1-2x_s\cos\psi_t+x_s^2}\;\frac{}{1-2y_s\cos\psi_t+y_s^2}\;d\psi_1\ldots d\psi_{\nu-1}$$

$$\frac{K}{2^{\nu^2-\nu}\prod[(1-x_i^2)(1-y_i^2)]}\times$$

$$\times\sum\pm\prod\int\frac{1}{(1-2x_\nu\cos\psi_j+x_\nu^2)(1-2y_\nu\cos\psi_j+y_\nu^2)}\times$$

$$\times\frac{(1-\cos^2\psi_j)\,d\psi_j}{(1-2x_{\lambda_j}\cos\psi_j+x_{\lambda_j}^2)(1-2y_{\mu_j}\cos\psi_j+y_{\mu_j}^2)}$$

$$\frac{K\pi}{2^{\nu^2-\nu}\prod[(1-x_i^2)(1-y_i^2)]}\times$$

$$\times\sum\pm\prod\left[\frac{1}{(1-x_\nu y_\nu)(1-x_\nu x_{\lambda_j})(1-x_\nu y_{\mu_j})}\times\right.$$

$$\left.\times\frac{1-x_\nu y_\nu x_{\lambda_j}y_{\mu_j}}{(1-y_\nu x_{\lambda_j})(1-y_\nu y_{\mu_j})(1-x_{\lambda_j}y_{\mu_j})}\right]$$

$$\frac{K\pi}{2^{\nu^2-\nu}\prod[(1-x_i^2)(1-y_i^2)]\prod'[(1-x_ix_\nu)(1-y_iy_\nu)](1-x_\nu y_\nu)^{\nu-1}}\times$$

$$\times\left|\frac{1-x_\nu y_\nu x_s y_t}{1\cdot\ -x_sy_t}\right.$$

$\prod'$ denoting a product from $i = 1$ to $i = \nu-1$. The determinant is equal to

$$x_\nu y_\nu+\frac{1-x_\nu y_1}{1-x_sy_t}\Big|$$

$$= (1-x_\nu y_\nu)^{\nu-2}\frac{\Delta'(x)\Delta'(y)}{\prod'(1-x_iy_j)}[1-x_\nu y_\nu+x_\nu y_\nu(1-\prod' x_iy_i)].$$

$$\frac{(1-x_\nu y_\nu)^{\nu-2}\Delta'(x)\Delta'(y)[1-\prod(x_iy_i)]}{\prod'(1-x_iy_j)}.$$

Hence $$J_2 = \frac{K'[1-\prod (x_i y_i)]}{L(x)L(y)\prod [(1-x_i^2)(1-y_i^2)]\prod (1-x_i y_j)}.$$

By putting $x_i = y_j = 0$, we see that $K' = 1$.

Combining this with equation (11.7; 1) we obtain

$$L(x)L(y)J = \frac{L(x)L(y)}{2h}\int \frac{1}{f_1\cdots f_\nu f_1'\cdots f_\nu'}\,dT = \frac{1}{\prod [(1-x_i^2)(1-y_i^2)]\prod (1-x_i y_j)}. \tag{11.7; 2}$$

Lastly, for $n = 2\nu+1$, we have

$$J = \frac{1}{2h}\int \frac{1}{f_1\cdots f_\nu f_1'\cdots f_\nu'}\,dT = \frac{1}{h}\int \frac{1}{f_1\cdots f_\nu f_1'\cdots f_\nu'}\,dS$$

$$= K\int_0^{2\pi}\cdots\int_0^{2\pi} \frac{\prod (1-\cos\phi_i)\Delta^2(\cos\phi_i)\,d\phi_1\dots d\phi_\nu}{f_1\cdots f_\nu f_1'\cdots f_\nu'}$$

For this case

$$f_i = (1-x_i)\prod_j (1-2x_i\cos\phi_j+x_i^2),$$

$$f_i' = (1-y_i)\prod_j (1-2y_i\cos\phi_j+y_i^2).$$

Hence we obtain

$$L(x)L(y)\Delta(x)\Delta(y)\prod [(1-x_i)(1-y_i)]J$$

$$= K\int_0^{2\pi}\cdots\int_0^{2\pi}\prod (1-\cos\phi_i)\frac{\Delta(x)L(x)\Delta(\cos\phi)}{\prod (1-2x_i\cos\phi_j+x_i^2)}\times$$

$$\times\frac{\Delta(y)L(y)\Delta(\cos\phi)}{\prod (1-2x_i\cos\phi_j+x_i^2)}\,d\phi$$

$$= \frac{K}{2^{\nu^2-\nu}}\int^{2\pi}\cdots\int^{2\pi}\prod (1-\cos\phi_i)\,\frac{}{1-2x_s\cos\phi_t+x_s^2}\times$$

$$\times\left|\frac{1}{1-2y_s\cos\phi_t+y_s^2}\right.\,d\phi_1\dots d\phi_\nu$$

$$= \frac{K}{2^{\nu^2-\nu}}\sum\pm\prod\int_0^{2\pi}\frac{(1-\cos\phi_j)\,d\phi_j}{(1-2x_{\lambda_j}\cos\phi_j+x_{\lambda_j}^2)(1-2y_{\mu_j}\cos\phi_j+y_{\mu_j}^2)}$$

$$= \frac{K\pi^\nu}{2^{\nu^2-\nu}}\sum\pm\frac{}{(1+x_s)(1+y_t)(1-x_s y_t)}$$

$$= \frac{K'}{\prod [(1+x_i)(1+y_i)]}\frac{\Delta(x)\Delta(y)}{\prod (1-x_i y_j)}.$$

We thus arrive at the same equation as for $n = 2\nu$, namely

$$L(x)L(y)J = \frac{L(x)L(y)}{2h}\int\frac{1}{f_1\cdots f_\nu f_1'\cdots f_\nu'}\,dT$$

$$= \frac{1}{\prod\,[(1-x_i^2)(1-y_i^2)]\prod\,(1-x_iy_j)}. \qquad (11.7;3)$$

## 11.8. The characters of the orthogonal group

Let $f(x) = 0$ be the characteristic equation of an orthogonal matrix $T$. If

$$1/f(x) = 1+\sum h_r x^r,$$

then clearly $h_r$ is a character, simple or compound, of the orthogonal group, for it is the spur of an induced matrix of $T$.

Let
$$\frac{1-x^2}{f(x)} = 1+\sum q_r x^r,$$

so that
$$\prod\left(\frac{1-x_i^2}{f_i}\right) = 1+\sum q_{\mu_1}q_{\mu_2}\cdots q_{\mu_\nu}x_1^{\mu_1}\cdots x_\nu^{\mu_\nu}.$$

Put $z_i = 1+x_i^2$. Then the determinant

$$|x_s^{\nu-1},\ x_s^{\nu-2}z_s,\ x_s^{\nu-3}z_s^2,\ \ldots,\ z_s^{\nu-1}|$$

$$= \prod x_i^{\nu-1}\Delta\left(\frac{z_i}{x_i}\right)$$

$$= \prod x_i^{\nu-1}\prod\left(\frac{1+x_i^2}{x_i}-\frac{1+}{x_j}\right)$$

$$= \prod\,(x_i-x_j)\prod\,(1-x_ix_j)$$

$$= \Delta(x)L(x).$$

Thus

$$\Delta(x)L(x)\prod\left(\frac{1-x_i^2}{f_i}\right) = [1+\sum q_{\mu_1}\cdots q_{\mu_\nu}x_1^{\mu_1}\cdots x_\nu^{\mu_\nu}]|x_s^{\nu-1},\ x_s^{\nu-2}z_s,\ \ldots,\ z_s^{\nu-1}|.$$

The coefficient of $x_1^{\lambda_1+\nu-1}x_2^{\lambda_2+\nu-2}\cdots x_\nu^{\lambda_\nu}$ in this expression is clearly

$$|q_{\lambda_s-s+1},\ q_{\lambda_s-s}+q_{\lambda_s-s+2},\ q_{\lambda_s-s-1}+q_{\lambda_s-s+3},\ \ldots,\ q_{\lambda_s-\nu-s+2}+q_{\lambda_s+\nu-s}|. \qquad (11.8;1)$$

THEOREM. *Corresponding to every partition* $(\lambda)$ *into not more than* $\nu$ *parts, there is a representation of the orthogonal group with character*

$$\chi^{(\lambda)} = |q_{\lambda_s-s+1},\ q_{\lambda_s-s}+q_{\lambda_s-s+2},\ \ldots,\ q_{\lambda_s-\nu-s+2}+q_{\lambda_s+\nu-s}|.$$

The determinant $\chi^{(\lambda)}$ is clearly a polynomial in the $h_r$ with positive or negative integral coefficients. It is thus a linear function of the characters of the orthogonal group with integral coefficients.

From the above we have

$$\Delta(x)L(x)\prod\left(\frac{1-x_i^2}{f_i}\right)=\sum\pm\chi^{(\lambda)}x_1^{\lambda_1+\nu-1}x_2^{\lambda_2+\nu-2}\dots x_\nu^{\lambda_\nu},$$

so that

$$L(x)\prod\left(\frac{1-x_i^2}{f_i}\right)=\sum\chi^{(\lambda)}\{x;\lambda\}. \qquad (11.8;2)$$

Similarly,

$$L(y)\prod\left(\frac{1-y_i^2}{f_i'}\right)=\sum\chi^{(\lambda)}\{y;\lambda\}.$$

Since from (11.7; 3)

$$L(x)L(y)\frac{1}{2h}\int\prod\left(\frac{1-x_i^2}{f_i}\right)\prod\left(\frac{1-y_i^2}{f_i'}\right)dT=1/\prod(1-x_iy_j),$$

we have

$$\frac{1}{2h}\int\sum\chi^{(\lambda)}\{x;\lambda\}\chi^{(\mu)}\{y;\mu\}\,dT=1+\sum\{x;\lambda\}\{y;\lambda\}.$$

It follows that

$$\left.\begin{aligned}\frac{1}{2h}\int\chi^{(\lambda)}(T)^2\,dT&=1,\\ \frac{1}{2h}\int\chi^{(\lambda)}(T)\chi^{(\mu)}(T)\,dT&=0,\quad(\lambda)\neq(\mu).\end{aligned}\right\} \qquad (11.8;3)$$

Hence the expressions $\chi^{(\lambda)}$, with a possible change of sign, are distinct simple characters of $D'$. We shall subsequently evaluate $\chi^{(\lambda)}(I)$, and since this will be seen to be a positive quantity, the change of sign is not required.

Corresponding to each character $\chi^{(\lambda)}$, which is not a double character, there is an associated character $\chi^{(\lambda)*}$. We shall show that these characters $\chi^{(\lambda)}$ and $\chi^{(\lambda)*}$ comprise the complete set of characters of $D'$.

### The double characters

For $n=2\nu+1$, the characters of $D'$ are also simple characters of $D$.

For $n=2\nu$, we have

$$\Delta(x)\Delta(y)\frac{1}{h}\int\frac{\prod(1-x_i^2)\prod(1-y_i^2)\,dS}{f_1\dots f_\nu f_1'\dots f_\nu'}=\frac{1+\prod(x_iy_i)}{\prod(1-x_iy_j)}.$$

Hence

$$\begin{aligned}\frac{1}{h}\int\sum\chi^{(\lambda)}\{x;\lambda\}\chi^{(\mu)}\{y;\mu\}\,dS&\\ &=[1+\sum\{x;\lambda\}\{y;\lambda\}][1+\prod(x_ry_r)]\\ &=[1+\sum\{x;\lambda\}\{y;\lambda\}][1+\{x;1^\nu\}\{y;1^\nu\}].\end{aligned}$$

Now $$\{x;\lambda\}\{x;1^{\nu}\} = \{x;\lambda_1+1,\lambda_2+1,\ldots,\lambda_{\nu}+1\}$$
and $$\{y;\lambda\}\{y;1^{\nu}\} = \{y;\lambda_1+1,\lambda_2+1,\ldots,\lambda_{\nu+1}\}.$$

It follows that

$$\frac{1}{h}\int \chi^{(\lambda)}\chi^{(\mu)}\,dS = 0,$$

$$\begin{aligned}\frac{1}{h}\int \chi^{(\lambda)^2}\,dS &= 1 \quad \text{for } \lambda_{\nu} = 0,\\ &= 2 \quad \text{for } \lambda_{\nu} \neq 0.\end{aligned}$$

Thus if $\lambda_{\nu} = 0$, $\chi^{(\lambda)}$ is a single character of $D'$ and a simple character of $D$. But if $\lambda_{\nu} \neq 0$, $\chi^{(\lambda)}$ is a double character of $D'$ and separates into two conjugate characters of $D$,

$$\chi^{(\lambda)}(S) = \chi_1^{(\lambda)}(S)+\tilde{\chi}_1^{(\lambda)}(S).$$

**The degree of the representation**

We shall now evaluate $\chi^{(\lambda)}(I)$, which is the degree of the representation corresponding to $\chi^{(\lambda)}$.

The characteristic equation of $I_n$ is

$$f(x) \equiv (1-x)^n = 0.$$

Hence

$$\begin{aligned}(1-x^2)/f(x) = (1-x^2)(1-x)^{-n} &= (1+x)(1-x)^{1-n}\\ &= 1+\sum q_r x^r.\end{aligned}$$

Clearly

$$q_a = \binom{n+a-2}{a}+\binom{n+a-3}{a-1} = \binom{n+a-2}{n-2}+\binom{n+a-3}{n-2}.$$

Substituting in the determinant for $\chi^{(\lambda)}$, and consecutively subtracting each column from the next, we obtain

$$\begin{aligned}\chi^{(\lambda)}(I) &= |q_{a_s},\ q_{a_s+1}+q_{a_s-1},\ q_{a_s+2}+q_{a_s-2},\ \ldots,\ q_{a_s+\nu-1}+q_{a_s-\nu+1}|\\ &= \left|\binom{n+a_s-2}{n-2}+\binom{n+a_s-3}{n-2},\ \binom{n+a_s-1}{n-2}+\binom{n+a_s-4}{n-2},\right.\\ &\quad \left.\binom{n+a_s}{n-2}+\binom{n+a_s-5}{n-2},\ \ldots,\ \binom{n+a_s+\nu-3}{n-2}+\binom{n+a_s-\nu-2}{n-2}\right|,\end{aligned} \tag{11.8;4}$$

where $a_s = \lambda_s-s+1$.

Denote by $T_i$ the transformation in which, beginning at the $i$th column, each column is subtracted from the next column. By making the consecutive transformations

$$T_1,\ T_2,\ T_2,\ T_3,\ T_3,\ \ldots,$$

and by making use of the formula

$$\binom{n+1}{r+1} = \binom{n}{r+1}+\binom{n}{r},$$

it is easily seen that we obtain the determinant

$$\binom{n+a_s-2}{n-2}+\binom{n+a_s-3}{n-2},\quad \binom{n+a_s-3}{n-4}+\binom{n+a_s-4}{n-4},$$
$$\binom{n+a_s-4}{n-6}+\binom{n+a_s-5}{n-6},\quad \ldots$$
$$\binom{n+a_s-1-t}{n-2t}+\binom{n+a_s-2-t}{n-2t} \qquad (11.8;5)$$

Consider this determinant as a function of $a_1$, $a_2,\ldots$, $a_\nu$, treated as independent variables. It is zero if $a_p = a_q$, and also if

$$a_p+a_q = 2-n,$$

for this would make two rows identical except for a possible change of sign in the latter case.

Also, for $n$ odd, it is zero if $2a_p = 2-n$.

If $n$ is even, then the determinant is divisible by

$$\prod_{p<q}[(a_p-a_q)(a_p+a_q+n-2)],$$

and if $n$ is odd, by

$$\prod_{p<q}[(a_p-a_q)(a_p+a_q+n-2)]\prod(2a_p+n-2).$$

In either case we obtain a polynomial in the $a_p$'s of degree $\nu(n-\nu-1)$, which is clearly the degree of the determinant. Any other factor is therefore numerical.

By comparing the coefficient of

$$a_1^{n-2}a_2^{n-4}\ldots a_\nu^{n-2\nu},$$

we obtain, for $n = 2\nu$,

$$\left.\begin{aligned}
&\chi^{(\lambda)}(I)\\
&\quad = 2^\nu\prod[(a_p-a_q)(a_p+a_q+n-2)]/[(n-2)!\,(n-4)!\ldots 2!],\\
&\text{and, for } n = 2\nu+1,\\
&\chi^{(\lambda)}(I)\\
&\quad = \prod[(a_p-a_q)(a_p+a_q+n-2)]\prod(2a_p+n-2)\div\\
&\qquad\qquad \div[(n-2)!\,(n-4)!\ldots 1!].
\end{aligned}\right\} \quad (11.8;6)$$

In one case this result is incorrect. For the case $n = 2\nu$, $\lambda_\nu = 0$, the last row of the determinant in (11.8; 4) consists of zeros except for the last term, which is literally $\binom{n-2}{n-2}+\binom{-1}{n-2}$. The term $\binom{-1}{n-2}$ should be zero, but formal calculation gives $+1$.

Hence for $n = 2\nu$, $\lambda_\nu = 0$, the value of $\chi^{(\lambda)}(I)$ is one-half of the value given in equation (11.8; 6).

### Completeness of the sets of characters obtained for $D$ and $D'$

We now show that the sets of characters obtained for $D$ and $D'$ form the complete sets. It is only necessary to show this for $D$, by reason of the relations we have proved connecting these characters with those of $D'$.

Let $\chi_1(S)$ be any character of $D$. Then $\chi_1(S)$ is clearly a class function of $D$. If it is not an even class function, there is a conjugate character $\tilde{\chi}_1(S)$. Then $\chi_1(S)+\tilde{\chi}_1(S) = \chi(S)$ is an even class function of $D$. It is thus an even symmetric function of the parameters of $S$, $\phi_1, \phi_2,..., \phi_\nu$, of period $2\pi$ in each parameter.

By Fourier analysis $\chi(S)$ can thus be expressed as the sum of a series of terms which are the monomial symmetric functions of the quantities

$$\cos\phi_1,\ \cos\phi_2,\ ...,\ \cos\phi_\nu.$$

Clearly $\chi(S)$ can be expressed linearly in terms of the coefficients in the expansion of

$$\prod\left(\frac{1-x_i^2}{f_i}\right) = 1+\sum q_{\alpha_1} q_{\alpha_2}...q_{\alpha_\nu} x_1^{\alpha_1} x_2^{\alpha_2}...x_\nu^{\alpha_\nu}.$$

Each character of $D'$, $\chi^{(\lambda)}(T)$ is expressible linearly in terms of these coefficients. The corresponding equations may be ordered so that they may be solved consecutively to express the coefficients $q_{\alpha_1} q_{\alpha_2}...q_{\alpha_\nu}$ linearly in terms of the characters of $D'$, $\chi^{(\lambda)}(T)$.

Thus $\chi(S)$ can be expressed linearly in terms of the given set of characters $\chi^{(\lambda)}(S)$. It follows from the orthogonal relations that the characters obtained, namely

$$\chi^{(\lambda)}(S),\quad \chi_1^{(\lambda)}(S),\quad \tilde{\chi}_1^{(\lambda)}(S)$$

form the complete set of characters of $D$.

## 11.9. Alternative forms for the characters of the orthogonal group (7)

We obtain first the expansions of six symmetric functions as sums of series of $S$-functions of ascending weight. The $S$-functions are those associated with the series

$$1/f(x) = 1/\prod (1-\alpha_r x) = 1+\sum \{n\}x^n.$$

The six expansions are as follows:

$$\prod (1-\alpha_i \alpha_j) = 1+\sum (-1)^{\frac{1}{2}p}\{\alpha\}, \qquad (11.9;1)$$

$$1/\prod (1-\alpha_i \alpha_j) = 1+\sum \{\beta\}, \qquad (11.9;2)$$

$$\prod (1-\alpha_i^2)\prod (1-\alpha_i \alpha_j) = 1+\sum (-1)^{\frac{1}{2}p}\{\gamma\}, \qquad (11.9;3)$$

$$1/\prod (1-\alpha_i^2)\prod (1-\alpha_i \alpha_j) = 1+\sum \{\delta\}, \qquad (11.9;4)$$

$$\prod (1-\alpha_i)\prod (1-\alpha_i \alpha_j) = 1+\sum (-1)^{\frac{1}{2}(p+r)}\{\epsilon\}, \qquad (11.9;5)$$

$$1/\prod (1-\alpha_i)\prod (1-\alpha_i \alpha_j) = 1+\sum \{\zeta\}, \qquad (11.9;6)$$

in which $p$ is the weight of the $S$-function, and $r$ is the rank of the partition. The extent of the summations in each case is as follows.

$\{\alpha\}$ is summed for all partitions which in Frobenius notation are of the form

$$\begin{pmatrix} a \\ a+1 \end{pmatrix}, \quad \begin{pmatrix} a, & b \\ a+1, & b+1 \end{pmatrix}, \quad \begin{pmatrix} a, & b, & c \\ a+1, & b+1, & c+1 \end{pmatrix}, \quad \ldots,$$

e.g. $1-\{1^2\}+\{2\,1^2\}-\{3\,1^3\}-\{2^3\}+\{4\,1^4\}+\{3\,2^2\,1\}-\ldots;$

$\{\beta\}$ is summed for all partitions such that there are an even number of parts of any given magnitude, e.g:

$$1+\{1^2\}+\{2^2\}+\{1^4\}+\{2^2\,1^2\}+\{3^2\}+\{1^6\}+\{4^2\}\ldots;$$

$\{\gamma\}$ is summed for all partitions which in Frobenius notation are of the form

$$\begin{pmatrix} a+1 \\ a \end{pmatrix}, \quad \begin{pmatrix} a+1, & b+1 \\ a, & b \end{pmatrix}, \quad \begin{pmatrix} a+1, & b+1, & c+1 \\ a, & b, & c \end{pmatrix}, \quad \ldots,$$

e.g. $1-\{2\}+\{3\,1\}-\{4\,1^2\}-\{3^2\}+\{5\,1^3\}+\{4\,3\,1\}-\ldots;$

$\{\delta\}$ is summed for all partitions into even parts only, e.g.

$$1+\{2\}+\{4\}+\{2^2\}+\{6\}+\{4\,2\}+\{2^3\}+\{8\}+\ldots;$$

$\{\epsilon\}$ is summed for all self-conjugate partitions, e.g.

$$1-\{1\}+\{2\,1\}-\{2^2\}-\{3\,1^2\}+\{3\,2\,1\}+\{4\,1^3\}-\{3^2\,2\}-\{4\,2\,1^2\}-\ldots,$$

and $\{\zeta\}$ is summed for all $S$-functions.

The methods of proof differ considerably in the different cases. In the first case we have

$$\begin{aligned} &\prod (1-\alpha_i\alpha_j) \prod_{i<j} (\alpha_i-\alpha_j) \\ &\qquad = \prod (1+\alpha_i^2)^{n-1} \prod \left(\frac{\alpha_i}{1+\alpha_i^2}-\frac{\alpha_j}{1+\alpha_j^2}\right) \\ &\qquad = |\alpha_s^{n-1},\ \alpha_s^{n-2}(1+\alpha_s^2),\ \alpha_s^{n-3}(1+\alpha_s^2)^2,\ \ldots,\ (1+\alpha_s^2)^{n-1}| \\ &\qquad = |\alpha_s^{n-1},\ \alpha_s^{n-2}+\alpha_s^n,\ \alpha_s^{n-3}+\alpha_s^{n+1},\ \ldots,\ 1+\alpha_s^{2n}|, \end{aligned}$$

by a simple transformation. Expressing this as a sum of $2^{n-1}$ determinants and removing the factor $\prod (\alpha_i-\alpha_j)$, we obtain

$$\begin{aligned} \prod (1-\alpha_i\alpha_j) = {} & \{0\}+\{0,2\}+\{0,0,4\}+\ldots \\ & +\{0,2,4\}+\{0,2,0,6\}+\ldots \\ & +\{0,0,4,6\}+\ldots \\ & +\{0,2,4,6\}+\{0,2,4,0,8\} \\ & +\ldots. \end{aligned}$$

Reducing the parts of these $S$-functions to descending order, we obtain (11.9; 1).

Again, multiplying the $s$th row of the above determinant by $(1+\alpha_s)$ and subtracting each column from the next, we obtain

$$\begin{aligned} &\prod (1+\alpha_i) \prod (1-\alpha_i\alpha_j) \prod (\alpha_i-\alpha_j) \\ &\qquad\qquad = |\alpha_s^{n-1}+\alpha_s^n,\ \alpha_s^{n-2}+\alpha_s^{n+1},\ \ldots,\ 1+\alpha_s^{2n-1}|. \end{aligned}$$

Expressing this determinant as a sum of $2^n$ determinants, and removing the factor $\prod (\alpha_i-\alpha_j)$, we are led to

$$\begin{aligned} \prod (1+\alpha_i) \prod (1-\alpha_i\alpha_j) = {} & \{0\}+\{1\}+\{0,3\}+\{0,0,5\}+\ldots \\ & +\{1,3\}+\{1,0,5\}+\{1,0,0,7\}+\ldots \\ & +\{0,3,5\}+\ldots \\ & +\{1,3,5\}+\ldots \\ & +\ldots. \end{aligned}$$

Reducing the parts in each $S$-function to descending order and changing the sign of the $\alpha_i$'s throughout gives (11.9; 5).

The expression

$$1/\prod (1-\alpha_i^2) \prod (1-\alpha_i\alpha_j)$$

is, from § 10.3, Theorem III, a generating function for the concomitants of a quadratic in $n$ variables. The irreducible concomitants† are of type $\quad (2),\ (2^2),\ (2^3),\ \ldots,\ (2^n),$

---

† See Young (75), Part II.

and thus the complete set of concomitants correspond to all partitions into even parts only. We obtain (11.9; 4).

The effect of replacing $f(x)$ by $1/f(-x)$ is to replace

$$\prod(1-\alpha_i^2)\prod(1-\alpha_i\alpha_j) \quad \text{by} \quad \prod(1-\alpha_i\alpha_j),$$

and to replace each $S$-function by the conjugate $S$-function. We may thus deduce (11.9; 2) from (11.9; 4), and (11.9; 3) from (11.9; 1).

Lastly,

$$\begin{aligned} 1/\prod(1-\alpha_i)\prod(1-\alpha_i\alpha_j) &= \prod(1+\alpha_i)/\prod(1-\alpha_i^2)\prod(1-\alpha_i\alpha_j) \\ &= [1+\sum\{1^r\}][1+\sum\{\delta\}]. \end{aligned}$$

Since every $S$-function can be obtained in one and only one way as a term in a product

$$\{1^r\}\{\delta\},$$

equation (11.9; 6) follows.

Denoting the character of the orthogonal group corresponding to the partition $(\lambda)$ by $[\lambda]$, we saw from (11.8; 2) in the last section that

$$\frac{L(x)\prod(1-x_i^2)}{\prod f_i} = 1+\sum[\lambda]\{x;\lambda\}.$$

Now $$1/\prod f_i = 1+\sum\{\lambda\}\{x;\lambda\},$$

$\{\lambda\}$ being an $S$-function of the characteristic roots of the matrix element of the group.

Hence

$$[L(x)\prod(1-x_i^2)][1+\sum\{\lambda\}\{x;\lambda\}] = 1+\sum[\lambda]\{x;\lambda\}.$$

Since

$$\begin{aligned} L(x)\prod(1-x_i^2) &= \prod(1-x_i^2)\prod(1-x_i x_j) \\ &= 1+\sum(-1)^{\frac{1}{2}p}\{x;\gamma\}, \end{aligned}$$

by substituting in the above equation and equating coefficients of $\{x;\lambda\}$ we obtain

I. $$[\lambda] = \{\lambda\}+\sum(-1)^{\frac{1}{2}p}g_{\gamma\eta\lambda}\{\eta\},$$

*summed for all $S$-functions of the set $\{\gamma\}$ such that $\{\lambda\}$ appears in a product $\{\gamma\}\{\eta\}$ with coefficient $g_{\gamma\eta\lambda}$, $(\gamma)$ being a partition of $p$.*

Similarly, from the equation

$$[L(x)\prod(1-x_i^2)]^{-1}[1+\sum[\lambda]\{x;\lambda\}] = 1+\sum\{\lambda\}\{x;\lambda\},$$

we obtain

II. $$\{\lambda\} = [\lambda]+\sum g_{\delta\eta\lambda}[\eta],$$

*summed for $S$-functions of the set $\{\delta\}$.*

These theorems give readily the reduction of the invariant matrices of orthogonal matrices as a direct sum of irreducible representations of the group, or, equivalently, the expression of the $S$-functions of the characteristic roots of a matrix element of the group as a sum of simple characters of the group; and, conversely, the expression of the simple characters of the group in terms of the $S$-functions.

As an example take $[\lambda] = [3^2\,2]$, and apply Theorem I. Corresponding to the partition expressed in Frobenius notation as

$$\begin{pmatrix} 2 & 1 \\ 1 & 0 \end{pmatrix}, \quad \text{we have} \quad \begin{aligned} \{\gamma\} &= \{2\}, & \sum g_{\gamma\eta\lambda}\{\eta\} &= \{3^2\}+\{3\,2\,1\},\\ \{\gamma\} &= \{3\,1\}, & \sum g_{\gamma\eta\lambda}\{\eta\} &= \{3\,1\}+\{2^2\},\\ \{\gamma\} &= \{3^2\}, & \sum g_{\gamma\eta\lambda}\{\eta\} &= \{2\}. \end{aligned}$$

Thus $\quad [3^2\,2] = \{3^2\,2\}-\{3^2\}-\{3\,2\,1\}+\{3\,1\}+\{2^2\}-\{2\}.$

Similarly, using Theorem II,

$$\text{for} \quad \begin{aligned} \{\delta\} &= \{2\}, & \sum g_{\delta\eta\lambda}[\eta] &= [3^2]+[3\,2\,1],\\ \{\delta\} &= \{2^2\}, & \sum g_{\delta\eta\lambda}[\eta] &= [3\,1]+[2\,1^2],\\ \{\delta\} &= \{2^3\}, & \sum g_{\delta\eta\lambda}[\eta] &= [1^2]. \end{aligned}$$

We thus have

$$\{3^2\,2\} = [3^2\,2]+[3^2]+[3\,2\,1]+[3\,1]+[2\,1^2]+[1^2].$$

The analysis of the direct product of two representations of the orthogonal group into irreducible representations can be effected quite easily by this means. Since the character of the direct product is the product of the characters, we have only to express the product of the characters as a sum of simple characters. The following example illustrates the method.

$$\begin{aligned} [2^2][2] &= [\{2^2\}-\{2\}][\{2\}-1]\\ &= \{4\,2\}+\{3\,2\,1\}+\{2^3\}-\{4\}-\{3\,1\}-\{2^2\}-\{2^2\}+\{2\}\\ &= [4\,2]+[4]+[3\,1]+[2^2]+[2]+[2]+1\\ &\qquad +[3\,2\,1]+[3\,1]+[2^2]+[2\,1^2]+[2]+[1^2]\\ &\qquad +[2^3]+[2^2]+[2]+1\\ &\qquad -[4]-[2]-1\\ &\qquad -[3\,1]-[2]-[1^2]\\ &\qquad -2[2^2]-2[2]-2\\ &\qquad +[2]+1\\ &= [4\,2]+[3\,2\,1]+[2^3]+[3\,1]+[2^2]+[2\,1^2]+[2]. \end{aligned}$$

A simpler method of calculation is to make a similar use of Theorems III and IV which follow. The formulae are then obtained for the special case $n = 2\nu$ for a matrix of negative determinant. But since such formulae are quite general, if they are true for such a special case they are true in the general case.

The characters of the orthogonal group can also be expressed in terms of the $S$-functions of the *variable characteristic roots only.*

If $n = 2\nu$, all the characteristic roots of a matrix $S$ of positive determinant are variable, and the case is covered by Theorems I and II.

For a matrix $U$ of negative determinant, however, we have

$$\begin{aligned} f(x) &= (1-x^2)\prod(1-e^{i\psi}x)(1-e^{-i\psi}x) \\ &= (1-x^2)f'(x), \quad \text{say.} \end{aligned}$$

We express the characters in terms of the $S$-functions $\{\lambda\}'$ of $1/f'(x)$.

Since $$\frac{L(x)\prod(1-x_i^2)}{\prod f(x_i)} = \frac{L(x)}{\prod f'(x_i)} = 1+\sum[\lambda]\{x;\lambda\},$$

and $$L(x) = 1+\sum(-1)^{\frac{1}{2}p}\{\alpha\},$$

we obtain

III. *For a matrix $U$ of negative determinant*

$$[\lambda] = \{\lambda\}'+\sum(-1)^{\frac{1}{2}p}g_{\alpha\eta\lambda}\{\eta\}',$$

*$\{\lambda\}'$ denoting an $S$-function of the variable characteristic roots $e^{\pm i\psi_1},\ldots,$ $e^{\pm i\psi_\nu}$.*

And from the equation

$$1/L(x) = 1+\sum\{x;\beta\},$$

IV. *If $n = 2\nu$, for a matrix $U$ of negative determinant*

$$\{\lambda\}' = [\lambda]+\sum g_{\beta\eta\lambda}[\eta].$$

As we have mentioned, these theorems give an easier method of analysing the direct product of two irreducible representations.

Thus for partitions into one part, since the set $\{\alpha\}$ contains no partitions into one part, we have, for $p > q$,

$$\begin{aligned} [p][q] &= \{p\}'\{q\}' \\ &= \{p+q\}'+\{p+q-1,1\}'+\{p+q-2,2\}'+\ldots+\{p,q\}'. \end{aligned}$$

For partitions into two parts, the set $\{\beta\}$ includes the $S$-functions $\{1^2\}$, $\{2^2\}$, $\{3^2\}$,.... Hence

$$\{p,q\}' = [p,q]+[p-1,q-1]+[p-2,q-2]+\ldots+[p-q].$$

Thus

$$\begin{aligned}[p][q] = [p+q] & \\ & +[p+q-1,1]+[p+q-2] \\ & +[p+q-2,2]+[p+q-3,1]+[p+q-4] \\ & + \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \\ & +[p,q]+[p-1,q-1]+...+[p-q].\end{aligned}$$

This is a generalized form of the Clebsch-Gordan formula.

For the case $n = 2\nu+1$ we put

$$f(x) = (1-\theta x)f'(x),$$

where $\theta = \pm 1$ is equal to the determinant of the matrix.

In a similar manner, using equations (11.9; 5), (11.9; 6), we arrive at

V. *For* $n = 2\nu+1$

$$[\lambda] = \{\lambda\}' + \sum (-1)^{\frac{1}{2}(p+r)} \theta^p g_{\epsilon\eta\lambda}\{\eta\}',$$

*$\theta$ being the determinant of the matrix element and $\{\lambda\}'$ denoting an S-function of the variable characteristic roots.*

VI. *For* $n = 2\nu+1$

$$\{\lambda\}' = [\lambda] + \sum (-\theta)^p g_{\zeta\eta\lambda}[\eta].$$

We can also express the characters of the orthogonal group in terms of the parametric angles. We obtain formulae which closely resemble that which expresses the $S$-function as a quotient of determinants (8).

We must consider the four cases separately, taking first the case of a matrix element $S$ of positive determinant, for $n = 2\nu$. We have

$$f(x) = \prod (1-2x\cos\phi_i+x^2).$$

We make use of the expansion

$$\frac{1-x^2}{1-2x\cos\phi+x^2} = 1+\sum 2x^r \cos r\phi,$$

and for convenience we put

$$\left.\begin{aligned} C_i^{(0)} &= 1, \\ C_i^{(r)} &= 2\cos r\phi_i \quad (r \neq 0). \end{aligned}\right\} \qquad (11.9; 7)$$

From Lemma III, § 11.7, we have

$$\frac{1}{1-2x_s\cos\phi_t+x_s^2} \qquad \frac{2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\phi)\Delta(x)L(x)}{\prod (1-2x_i\cos\phi_j+x_i^2)}.$$

Hence

$$\begin{aligned}\frac{2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\phi)\Delta(x)L(x)\prod(1-x_i^2)}{\prod(1-2x_i\cos\phi_j+x_i^2)} &= \frac{1-x_s^2}{|1-2x_s\cos\phi_t+x_s^2|}\\ &= |\sum C_t^{(r)}x_s^r|\\ &= \sum |C_t^{(\lambda_s+\nu-s)}|\,.\,|x_s^{\lambda_t+\nu-t}|\\ &= \sum |C_t^{(\lambda_s+\nu-s)}|\{x;\lambda\}\Delta(x).\end{aligned}$$

Thus, from (11.8; 2),

$$\begin{aligned}1+\sum[\lambda]\{x;\lambda\} &= \frac{L(x)\prod(1-x_i^2)}{\prod f(x_i)}\\ &= \frac{\sum|C_t^{(\lambda_s+\nu-s)}|\{x;\lambda\}}{2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\phi)},\end{aligned}$$

and by comparing coefficients of $\{x;\lambda\}$,

$$\begin{aligned}[\lambda] &= |C_t^{(\lambda_s+\nu-s)}|/2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\phi)\\ &= |C_t^{(\lambda_s+\nu-s)}|/|C_t^{(\nu-s)}|.\end{aligned}$$

VII. *For $n = 2\nu$, and for a matrix element of positive determinant,*

$$[\lambda] = |C_t^{(\lambda_s+\nu-s)}|/|C_t^{(\nu-s)}|.$$

But for a matrix $U$ of negative determinant

$$f(x) = (1-x^2)\prod(1-2x\cos\psi_i+x^2).$$

In this case we need the expansion

$$\frac{1}{1-2x\cos\theta+x^2} = \frac{1}{2i\sin\theta}\sum_{r=1}^{\infty} 2x^{r-1}\sin r\theta.$$

Hence, from the equation

$$\frac{1}{1-2x_s\cos\psi_t+x_s^2} \qquad \frac{2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\psi)\Delta(x)L(x)}{\prod(1-2x_i\cos\psi_j+x_i^2)}$$

we obtain

$$\begin{aligned}1+\sum[\lambda]\{x;\lambda\} &= \frac{L(x)\prod(1-x_i^2)}{\prod f_i} = \frac{L(x)}{\prod(1-2x_i\cos\psi_j+x_i^2)}\\ &\quad \frac{1}{1-2x_s\cos\psi_t+x_s^2} \div 2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\psi)\Delta(x)\\ &\quad \frac{1}{2\sin\psi_t}\sum 2x^{r-1}\sin r\psi_t \div 2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\psi)\Delta(x)\\ &= \frac{\sum|S_t^{(\lambda_s+\nu-s+1)}|\{x;\lambda\}}{\prod 2\sin\psi_i\, 2^{\frac{1}{2}(\nu^2-\nu)}\Delta(\cos\psi)}\\ &= \sum\frac{|S_t^{(\lambda_s+\nu-s+1)}|}{|S_t^{(\nu-s+1)}|}\{x;\lambda\},\end{aligned}$$

where

$$S_i^{(j)} = 2\sin j\psi_t. \qquad (11.9; 8)$$

VIII. *For $n = 2\nu$, and for a matrix element of negative determinant,*

$$[\lambda] = |S_i^{(\lambda_s+\nu-s+1)}|/|S_i^{(\nu-s+1)}|.$$

For $n = 2\nu+1$ we have for an element of positive determinant,

$$f(x) = (1-x)\prod(1-2x\cos\phi_i+x^2).$$

We must use the expansion of

$$\frac{1-x^2}{(1-x)(1-2x\cos\phi+x^2)} = \frac{1+x}{(1-xe^{i\phi})(1-xe^{-i\phi})}$$

$$= \frac{1}{(1-e^{i\phi})(1-xe^{-i\phi})}+\frac{1}{(1-e^{-i\phi})(1-xe^{i\phi})}$$

$$= \sum \frac{e^{(r+\frac{1}{2})i\phi}-e^{-(r+\frac{1}{2})i\phi}}{e^{\frac{1}{2}i\phi}-e^{-\frac{1}{2}i\phi}}x^r$$

$$= \frac{1}{2\sin\frac{1}{2}\phi}\sum 2x^r\sin(r+\tfrac{1}{2})\phi.$$

But for an element of negative determinant we obtain

$$\frac{1-x}{(1-xe^{i\psi})(1-xe^{-i\psi})} = \frac{1}{2\cos\frac{1}{2}\psi}\sum 2x^r\cos(r+\tfrac{1}{2})\psi.$$

By a procedure similar to that adopted for the case $n = 2\nu$ we obtain

IX. *If $n = 2\nu+1$, then for a matrix $S$ of positive determinant*

$$[\lambda] = \frac{|S_i^{(\lambda_s+\nu-s+\frac{1}{2})}|}{|S_i^{(\nu-s+\frac{1}{2})}|}$$

*and for a matrix $U$ of negative determinant*

$$[\lambda] = \frac{|C_i^{(\lambda_s+\nu-s+\frac{1}{2})}|}{|C_i^{(\nu-s+\frac{1}{2})}|},$$

*where*

$$S_i^{(j)} = 2\sin j\phi_i,$$
$$C_i^{(j)} = 2\cos j\psi_i.$$

The proof for the element $U$ of negative determinant could have been obtained from that for the element $S$ of positive determinant by putting

$$\phi_i = \pi+\psi_i.$$

## 11.10. The difference characters of the rotation group

For the case $n = 2\nu$, the characters $[\lambda]$ of the orthogonal group, for which $\lambda_\nu \neq 0$, are double characters, and separate into two conjugate characters of the corresponding rotation group. So far we have found only the values of the double character, and not the values of the separate conjugate characters.

Denote the difference between the two conjugate characters of the rotation group, corresponding to the partition $(\lambda)$, by $[\lambda]'$. The two conjugate characters of the rotation group will thus be

$$\tfrac{1}{2}\{[\lambda]+[\lambda]'\} \quad \text{and} \quad \tfrac{1}{2}\{[\lambda]-[\lambda]'\}.$$

$[\lambda]'$ is called a *difference character* of the rotation group (9).

If the two conjugate characters are real, i.e. if $\nu$ is even, then $[\lambda]'$ will be real. If the conjugate characters are complex, i.e. if $\nu$ is odd, then $[\lambda]'$ will be pure imaginary.

From the orthogonal properties of the characters it is clearly seen that the difference characters satisfy the equations

$$\left.\begin{aligned} &\frac{1}{h}\int [\lambda]'^2\, dS = (-1)^\nu, \\ &\int [\lambda]'[\mu]'\, dS = 0, \quad (\lambda) \neq (\mu). \end{aligned}\right\} \tag{11.10;1}$$

We obtain first the difference character $[1^\nu]'$.

The character $[1^\nu]$ of the orthogonal group is equal to the $S$-function $\{1^\nu\}$ of the characteristic roots

$$e^{i\phi_1},\ e^{-i\phi_1},\ e^{i\phi_2},\ \ldots,\ e^{i\phi_\nu},\ e^{-i\phi_\nu},$$

which is equal to the sum of the products of these roots $\nu$ at a time.

The interchange of a pair of characteristic roots $e^{i\phi_r}$ and $e^{-i\phi_r}$ has the effect of interchanging pairs of conjugate classes, and thus pairs of conjugate characters. The difference character is thereby changed in sign, and $[\lambda]'$ is an odd function of each parameter $\phi_r$.

Clearly, then, the only products of $\nu$ characteristic roots which appear in the difference character $[1^\nu]'$ are those which involve all the $\nu$ parameters. Thus the difference character is given by

$$[1^\nu]' = \sum \pm e^{i(\pm\phi_1\pm\phi_2\pm\ldots\pm\phi_\nu)},$$

summed for all combinations of the alternative signs in the index, the prefixed sign being plus or minus according as the number of negative signs in the index is even or odd.

Hence
$$\begin{aligned}[1^\nu]' &= \prod (e^{i\phi_r}-e^{-i\phi_r}) \\ &= \prod (2i\sin\phi_r).\end{aligned}$$

I. $$[1^\nu]' = \prod (2i\sin\phi_r).$$

Now since the product of two characters of a group is a character, simple or compound, the product of a difference character of the rotation group and a character of the corresponding orthogonal group will give a difference character, simple or compound, of the rotation group.

Hence, for every partition $(\lambda)$ into not more than $\nu$ parts, the product

$$[1^\nu]'\{\lambda\}$$

is a difference character, simple or compound, of the rotation group.

The simple difference characters are obtained from these by the following theorem (**10**).

II. *The values of the difference characters of the rotation group are given by*

$$[\lambda_1+1,\lambda_2+1,\ldots,\lambda_\nu+1]' = [1^\nu]'[\{\lambda\}+\sum(-1)^{\frac{1}{2}p}g_{\alpha\eta\lambda}\{\eta\}]$$

*summed for those S-functions* $\{\alpha\}$ *defined by* (11.9; 3), *such that* $\{\lambda\}$ *appears with coefficient* $g_{\alpha\eta\lambda}$ *in the product* $\{\alpha\}\{\eta\}$.

Clearly the right-hand side of this equation is a linear function of the difference characters with positive or negative integral coefficients. Denoting this expression by $\xi^{(\lambda)}$, we show first that

$$\frac{1}{h}\int \xi^{(\lambda)^2}\,dS = (-1)^\nu,$$

which proves that $\xi^{(\lambda)}$ is a simple difference character. When this is proved, consideration of the highest powers of $e^{i\phi_1}, e^{i\phi_2},\ldots, e^{i\phi_\nu}$, taken in this order, which appear in the difference character, clearly indicates that it is the appropriate difference character for the partition shown on the left-hand side of the equation.

If $\phi_1,\ldots, \phi_\nu$ are the parametric angles of the given rotation $S$, let $U$ denote the orthogonal matrix of negative determinant of the group $G$ of orthogonal matrices of order $(2\nu+2)^2$, of which the parametric angles are $\phi_1,\ldots, \phi_\nu$ also.

Let $$h' = \int\cdots\int \prod(\sin^2\phi_r)\Delta(\cos\phi_x)\,d\phi_1\ldots d\phi_\nu.$$

Then, from Theorem III, § 11.9,

$$[\lambda]_1 = \{\lambda\}+\sum(-1)^{\frac{1}{2}p}g_{\alpha\eta\lambda}\{\eta\}$$

is a simple character of $G$, and

$$\frac{1}{h'}\int\cdots\int [\lambda]_1^2\prod(\sin^2\phi_r)\Delta(\cos\phi_r)\,d\phi_1\ldots d\phi_\nu = 1.$$

Hence

$$\frac{1}{h}\int\cdots\int \xi^{(\lambda)2}\,dS = \frac{1}{h}\int\cdots\int [\lambda]_1^2 \prod (-4\sin^2\phi_r)\Delta(\cos\phi_r)\,d\phi_1\ldots d\phi_\nu$$

$$= (-1)^\nu 2^\nu h'/h.$$

By taking the special case $[\lambda]_1 = 1$ it is clear that $2^\nu h'/h = 1$, and the proof of the theorem is complete.

By comparison with the characters of $U$ there is no difficulty in the deduction of the following theorems from Theorems IV and VIII, §11.9:

III. $[1^\nu]'\{\lambda\} = [\lambda_1+1,\ldots,\lambda_\nu+1]' + \sum g_{\beta\eta\lambda}[\eta_1+1,\ldots,\eta_\nu+1]'$.

IV. $[\lambda]' = |S_t^{(\lambda_s+\nu-s)}|/|C_t^{(\nu-s)}|$.

## 11.11. The spin representations of the orthogonal group (11)

If the group $D'$ of orthogonal matrices of degree $n$ has a representation of degree $N$, this implies that the group of unitary matrices of degree $N$ has a subgroup simply isomorphic with $D'$.

It may happen that, while this is not the case, yet the group of unitary matrices has a subgroup $G$ possessing a self-conjugate subgroup $\Gamma$ of finite order $p$, and the quotient group $G/\Gamma$ is simply isomorphic with $D'$. In this event, $D'$ will have a $p$-valued representation of degree $N$, i.e. to each matrix of $D'$ there correspond $p$ matrices of degree $N$ such that the product of two matrices corresponding to $S$ and $T$ respectively is one of the $p$ matrices corresponding to $ST$.

This is not strictly a representation of the group, which is by definition single-valued, but it will be called a *$p$-valued representation.* A one-valued representation, by contrast, will be called a *true representation.*

Given a true representation $\mathscr{G}$ of degree $N$, a $p$-valued representation may be obtained by taking $G$ as the direct product of $\mathscr{G}$ and the finite group of order $p$ which consists of the scalar $p$th roots of unity. Also, given a $p$-valued representation, a $pr$-valued representation may be obtained in the same manner. Such representations are, however, trivial, and it is our purpose to examine the $p$-valued representations which cannot be so generated.

We shall show that non-trivial $p$-valued representations exist for every degree of $D'$, for $p = 2$, but for no other value of $p$.

We may suppose that the representation of degree $N$ of $G$ is irreducible. Hence any matrix which commutes with every matrix of the representation is a scalar multiple of the unit matrix. Thus the elements of $\Gamma$ are scalar multiples of the unit matrix.

The scalar multipliers, by their nature, must be $p$th roots of unity. We will show that they are also real. It will follow that the only possible values are $\pm 1$, and we must have $p = 2$.

The $p$ elements of $\Gamma$ correspond to the identical element of $D'$. Of the $p$ elements of $G$ which correspond to an infinitesimal rotation $S$, exactly one will differ infinitesimally from the unit matrix $I_N$. Denote this by $Z$. Since $S$ and $S^{-1}$ are conjugate in $D'$, clearly $Z$ and $Z^{-1}$ will be conjugate in $G$. It follows that the character of $Z$ is real.

Denoting by $D$ the elements of $D'$ with positive determinant, since every element of $D$ can be generated by infinitesimal rotations, there is a matrix of $G$ with real character corresponding to every element of $D$.

Ignoring the trivial case in which $G$ has a subgroup which gives a $q$-valued representation of $D'$, for $q < p$, we may assume that the characters of all elements of $G$ which correspond to elements of $D$ have real characters. This applies especially to the elements of $\Gamma$. It follows that the scalar multipliers are real, and equal to $\pm 1$.

I. *If a $p$-valued representation of the group of orthogonal matrices of degree $n$ is not trivial, then $p = 2$.*

We show next that two-valued representations exist for all values of $n$. Such representations are called *spin representations*. The set of spurs of the corresponding matrices is called a *spin character*.

The group $G$ which is doubly isomorphic with $D'$ is called the *covering group*.

We consider first the special case $n = 3$. The close connexion between quaternions and three-dimensional rotations has been well known since the discovery of the former by Hamilton.[†] In particular, there is an isomorphism between unimodular quaternions and three-dimensional rotations.

Let $i, j, k$ have their usual quaternion significance, so that

$$i^2 = j^2 = k^2 = -1, \qquad ij = -ji = k,$$

$$jk = -kj = i, \qquad ki = -ik = j,$$

and $$(a+bi+cj+dk)^{-1} = (a-bi-cj-dk)/(a^2+b^2+c^2+d^2).$$

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[†] See W. K. Hamilton, *Lectures on Quaternions* (Dublin, 1853).

With a given point in three-space with rectangular Cartesian coordinates $(x_1, x_2, x_3)$ we associate the vector (i.e. quaternion with real part equal to zero)

$$X = ix_1+jx_2+kx_3.$$

Then, if

$$A = a_0+ia_1+ja_2+ka_3,$$

and $\sum a_r^2 = 1$, the transformation

$$X' = A^{-1}XA,$$

where

$$X' = ix_1'+jx_2'+kx_3',$$

corresponds to a rotation through an angle $2\cos^{-1}a_0$ about the line

$$\frac{x_1}{a_1} = \frac{x_2}{a_2} = \frac{x_3}{a_3}.$$

This result is obvious for the special case $A = \cos\theta+i\sin\theta$, for we have

$$\begin{aligned} X' &= (\cos\theta-i\sin\theta)(x_1 i+x_2 j+x_3 k)(\cos\theta+i\sin\theta) \\ &= x_1 i+(x_2\cos 2\theta+x_3\sin 2\theta)j+(x_3\cos 2\theta-x_2\sin 2\theta)k. \end{aligned}$$

The general case may be transformed into this case by a change of axes, or may be verified directly.

The only unimodular quaternions which commute with $X$ are the real numbers $\pm 1$. Thus to each rotation $X \rightarrow X'$ there correspond exactly two unimodular quaternions $\pm A$ such that

$$X' = A^{-1}XA,$$

$$X' = (-A)^{-1}X(-A).$$

We have thus found a two-valued representation of the rotation group by the group of unimodular quaternions.

But the quaternions

$$A = a_0+ia_1+ja_2+ka_3$$

are simply isomorphic with the two-rowed complex matrices

$$A' = \begin{bmatrix} a_0+ia_1 & a_2+ia_3 \\ -a_2+ia_3 & a_0-ia_1 \end{bmatrix}. \tag{11.11;1}$$

We thus obtain a spin representation of the rotation group of degree 3. Since the full orthogonal group is the direct product of the rotation group and the group $(I, -I)$, we have also a spin representation of the orthogonal group.

If the parametric angle of a rotation is $\phi$, then

$$2\cos^{-1}a_0 = \phi,$$

$$a_0 = \cos\tfrac{1}{2}\phi.$$

The character of the spin representation is the spur of $A'$, which is

$$2a_0 = 2\cos\tfrac{1}{2}\phi.$$

II. *There is a spin representation of the orthogonal group of degree* 3 *with character* $\pm 2\cos\frac{1}{2}\phi$.

For an orthogonal matrix of negative determinant with characteristic roots

$$-1,\ e^{i\psi},\ e^{-i\psi},$$

since

$$\psi = \pi+\phi,$$

the character can be expressed as

$$\pm 2\cos\tfrac{1}{2}\phi = \pm 2\sin\tfrac{1}{2}\psi. \qquad (11.11;2)$$

We next obtain the characters of a spin representation of degree $2^\nu$ for the general case of the group of orthogonal matrices of degree $n = 2\nu$, or $n = 2\nu+1$. This is called the *basic spin representation.*

Consider a set of $n$ anticommuting quantities (**12**)

$$M_1,\ M_2,\ \ldots,\ M_n$$

such that their squares are all equal, and equal to $\pm 1$, i.e.

$$\left.\begin{aligned} M_p M_q &= -M_q M_p \quad (p \neq q), \\ M_p^2 &= M_q^2 = \pm 1. \end{aligned}\right\} \qquad (11.11;3)$$

If now we make a linear transformation amongst these quantities,

$$M_p' = \sum a_{pq} M_q,$$

then, clearly, the necessary and sufficient condition that the set of matrices

$$M_1',\ M_2',\ \ldots,\ M_n'$$

also satisfy the equations (11.11;3) is that $[a_{st}]$ is an orthogonal matrix.

If $n = 2\nu$, the quantities $M_1, M_2,\ldots, M_n$ are taken as algebraically independent.

If $n = 2\nu+1$, then $M_1, M_2,\ldots, M_{2\nu}$ are taken as algebraically independent, but we put

$$M_n = M_1 M_2 \ldots M_{2\nu}$$

or the same quantity multiplied by a scalar $\sqrt{(-1)}$ according to which of these gives $M_n^2 = M_1^2$.

In either case we have $2\nu$ algebraically independent quantities $M_1, M_2,..., M_{2\nu}$. Adding an identical element $I$ and an element $-I$ which corresponds to $-1$, these quantities generate a group of order $2^{2\nu+1}$ of which the general element is

$$\pm M_1^{\alpha_1} M_2^{\alpha_2} ... M_{2\nu}^{\alpha_{2\nu}} \quad (\alpha_r = 0 \text{ or } 1).$$

We now examine the matrix representations of this group.

There are clearly $2^{2\nu}+1$ classes of the group, which consist of the identity element $I$, $-I$, and $2^{2\nu}-1$ other classes each of two elements.

The quotient group by the subgroup $(I, -I)$ is clearly Abelian and possesses $2^{2\nu}$ characters of degree 1. Hence, from § 9.6, the group generated by $M_1,..., M_{2\nu}$ has $2^{2\nu}$ characters of degree 1, satisfying

$$\chi(I) = \chi(-I) = 1.$$

Since the number of characters of the group is equal to the number of classes, there is but one other character, which from the orthogonal properties must be of degree $2^\nu$.

There is thus one representation only of the group which discriminates $I$ from $-I$, and this is of degree $2^\nu$.

In this representation let each element $M_i$ be represented by $B_i$, and each $M_i'$ by $B_i'$. Since the quantities $M_i'$ are simply isomorphic with the quantities $M_i$, there is clearly a representation in which each matrix $M_i$ is represented by $B_i'$. Since there is only one independent representation of the group which distinguishes $I$ from $-I$, this must be equivalent to the first representation, i.e. there is a matrix $T(A)$ such that

$$T(A)^{-1} B_i' T(A) = B_i.$$

The set of matrices $T(A)$ clearly form a spin representation of the orthogonal group.

This basic spin representation of degree $2^\nu$ is denoted by $\Delta$. To find the corresponding character which we denote by $\chi(\Delta)$, we construct a specific representation of the group by means of the direct product of $\nu$ quaternion algebras. By substituting a two-rowed matrix representation of each quaternion algebra we thus obtain a specific matrix representation of degree $2^\nu$.

Let $i_r, j_r, k_r$ $(1 \leqslant r \leqslant \nu)$ be the basic elements of $\nu$ independent quaternion algebras. By allowing elements of different algebras to commute, the products of these quantities form the basal elements of an algebra which is the direct product of the $\nu$ quaternion algebras.

Then the $2\nu$ elements

$$i_1,\ j_1,\ k_1 i_2,\ k_1 j_2,\ k_1 k_2 i_3,\ k_1 k_2 j_3,\ \ldots,\ k_1 k_2 \ldots k_{\nu-1} j_\nu$$

are anticommuting. Adding a factor which is the scalar $\sqrt{(-1)}$ when necessary to ensure that the square of each element is $\pm 1$ as required, we obtain a representation of $M_1, M_2, \ldots, M_\nu$ as a direct product of $\nu$ quaternion algebras.

The matrix representation (11.11; 1) of each quaternion algebra then gives the required matrix representation of the group of degree $2^\nu$.

The matrix which corresponds to the direct product of $\nu$ quaternions of the form

$$\prod_{r=1}^{\nu} (\cos\tfrac{1}{2}\phi_r + k_r \sin\tfrac{1}{2}\phi_r)$$

is clearly a rotation with parametric angles $\phi_1, \phi_2, \ldots, \phi_\nu$. The matrix representation of

$$\cos\tfrac{1}{2}\phi_r + k_2 \sin\tfrac{1}{2}\phi_r$$

is clearly

$$\begin{bmatrix} \cos\tfrac{1}{2}\phi_r, & i\sin\tfrac{1}{2}\phi_r \\ i\sin\tfrac{1}{2}\phi_r, & \cos\tfrac{1}{2}\phi_r \end{bmatrix},$$

and the spur is $2\cos\tfrac{1}{2}\phi_r$.

Hence, the spur of the direct product being the product of the spurs, the corresponding character of the spin representation of the orthogonal group is

$$\prod (2\cos\tfrac{1}{2}\phi_r).$$

For an orthogonal matrix of negative determinant, the two cases $n = 2\nu$ and $n = 2\nu+1$ differ.

For $n = 2\nu+1$ we put $U_0 = -I$, and the two matrices $S$ and $U = U_0 S$ have the same representation and the same character.

For $n = 2\nu$ we replace $\phi_1, \phi_2, \ldots, \phi_{\nu-1}$ by $\psi_1, \psi_2, \ldots, \psi_{\nu-1}$, and the $\nu$th quaternion

$$\cos\tfrac{1}{2}\phi_\nu + k_\nu \sin\tfrac{1}{2}\phi_\nu$$

is replaced by $j_\nu$, which corresponds to the orthogonal matrix

$$\begin{bmatrix} 1, & 0 \\ 0, & -1 \end{bmatrix}.$$

The matrix representation of $j_\nu$ is

$$\begin{bmatrix} 0, & 1 \\ -1, & 0 \end{bmatrix},$$

and the spur of this matrix is zero. Thus for $n = 2\nu$ the spin character of every orthogonal matrix of negative determinant is zero.

III. *There is a basic spin representation* $\Delta$ *of degree* $2^\nu$ *of the orthogonal group of degree* $n = 2\nu$ *or* $n = 2\nu+1$. *The value of the corresponding spin character is given by*

$$\chi(\Delta) = \prod (2\cos\tfrac{1}{2}\phi_r),$$

*except in the case of a matrix element of negative determinant for* $n = 2\nu$, *in which case*
$$\chi(\Delta) = 0.$$

Before proving that $\Delta$ is irreducible, and constructing other spin representations, we first express $\chi(\Delta)^2$ as a sum of simple characters. This gives the analysis into irreducible representations of the direct product of $\Delta$ with itself.

We have

$$\chi(\Delta)^2 = \prod (2\cos\tfrac{1}{2}\phi_r)^2$$
$$= \prod (1+e^{i\phi_r})(1+e^{-i\phi_r}) = \sum_{r=0}^{2\nu} \{1^r\}',$$

$\{\lambda\}'$ denoting an $S$-function of the variable characteristic roots.

For $n = 2\nu$, for a rotation $S$, we have

$$\{\lambda\}' = \{\lambda\},$$

the $S$-function of all characteristic roots, and for $0 \leqslant r \leqslant \nu$,

$$\{1^r\} = [1^r];$$

while for $\nu < r \leqslant 2\nu$, $\qquad \{1^r\} = [1^{n-r}]^*, \qquad$ (11.11; 4)

the star denoting the associated character, i.e. the character obtained by changing the sign of the characteristic of each matrix of negative determinant.

Thus $$\chi(\Delta)^2 = \sum_{r=0}^{\nu-1}\{[1^r]+[1^r]^*\}+[1^\nu].$$

This expression is clearly self-associated, and is, correctly, zero for every matrix of negative determinant.

For $n = 2\nu+1$, however, $\{\lambda\}'$ and $\{\lambda\}$ are respectively the $S$-functions associated with

$$1/\prod [(1-ze^{i\phi})(1-ze^{-i\phi})]$$

and $$1/(1-z)\prod [(1-ze^{i\phi})(1-ze^{-i\phi})].$$

Thus $$\{\lambda\} = \{\lambda\}'+\sum g_{\mu\eta\lambda}\{\eta\}',$$

where $\{\mu\}$ is summed for the partitions $\{1\}$, $\{2\}$, $\{3\}$,....

Hence $$\chi(\Delta)^2 = \sum_{0}^{2\nu}\{1^r\}' = 1+\sum_{1}^{\nu}\{1^{2r}\},$$

As for the case $n = 2\nu$, we have, for $0 \leqslant 2r \leqslant \nu$,

$$\{1^{2r}\} = [1^{2r}],$$

but for $\nu < 2r \leqslant 2\nu$,

$$\{1^{2r}\} = [1^{n-2r}]^* = [1^{2\nu-2r+1}]^*.$$

IV. *For* $n = 2\nu$,

$$\chi(\Delta)^2 = \sum_0^{2\nu} \{1^r\} = \sum_0^{\nu-1} \{[1^r]+[1^r]^*\}+[1^\nu],$$

*and for* $n = 2\nu+1$,

$$\chi(\Delta)^2 = \sum_0^{\nu} \{1^{2r}\} = \sum [1^{2r}]+\sum [1^{2r+1}]^*.$$

Since $\chi(\Delta)^2$ includes the character $\chi^{(0)}$ exactly once, it follows that $\Delta$ is irreducible, for

$$\int [\lambda]\, dT = 0$$

for every other character, and thus the mean value of $\chi(\Delta)^2$ is the coefficient of $\chi^{(0)}$ which is unity.

The fact that $\Delta$ is strictly a representation of the covering group $G$, rather than of the orthogonal group $D'$, will not affect the mean value of $\chi(\Delta)^2$, as for a given orthogonal matrix the two values of $\chi(\Delta)$ are equal in magnitude but opposite in sign, and give the same value for $\chi(\Delta)^2$. Thus the mean value of $\chi(\Delta)^2$ is unaffected.

V. $\Delta$ *is an irreducible spin representation of the orthogonal group.*

It follows immediately that for $n = 2\nu+1$, $\Delta$ gives an irreducible spin representation of the rotation group.

For $n = 2\nu$, however, since $\Delta$ is a self-associated representation of the orthogonal group, i.e. $\chi(\Delta) = 0$ for every matrix of negative determinant, clearly the mean value of $\chi(\Delta)^2$ for the rotation group is 2, and $\Delta$ is reducible and equivalent to the direct sum of two conjugate spin representations.

We can find the characters of these conjugate spin representations by the same method as that adopted to prove Theorem I, § 11.10.

VI. *If* $n = 2\nu$, $\Delta$ *gives a reducible spin representation of the rotation group, being equivalent to the direct sum of two conjugate spin representations* $\Delta_1$ *and* $\Delta_2$ *with characters*

$$2^{\nu-1}[\prod (\cos\phi_r) \pm \prod (i \sin\phi_r)].$$

Besides the basic spin representation $\Delta$, there is a series of spin representations corresponding to all partitions into not more than $\nu$ parts. We now find the characters of these.

Clearly, from their nature, the direct product of a spin representation and a true representation will give a spin representation of the orthogonal group. Thus if $\{\lambda\}$ is an $S$-function of the characteristic roots of the orthogonal matrix,

$$\chi(\Delta)\{\lambda\}$$

will be the character of a spin representation, simple or compound. We obtain the simple spin characters from these by the use of the orthogonal relations.

It is convenient to adopt the following nomenclature for the spin characters. Each true representation is associated with a partition $(\lambda)$, i.e. with a set of $\nu$ integers $\lambda_1, \lambda_2,..., \lambda_\nu$. Each spin representation is associated with a set of $\nu$ numbers each of which is half an odd integer.

The basic spin representation $\Delta$ is associated with the set of numbers $((\frac{1}{2})^\nu)$, and the corresponding character is written

$$\chi(\Delta) = [(\tfrac{1}{2})^\nu].$$

This is consistent with the fact that the principal representation which appears in the direct product of $\Delta$ with itself is associated with the partition $(1^\nu)$.

The principal spin representation which appears in the direct product of $\Delta$ and the true representation associated with the partition $(\lambda)$, is associated with the sequence

$$(\lambda_1+\tfrac{1}{2}, \lambda_2+\tfrac{1}{2},..., \lambda_\nu+\tfrac{1}{2}),$$

and the corresponding character is written

$$[\lambda_1+\tfrac{1}{2}, \lambda_2+\tfrac{1}{2},..., \lambda_\nu+\tfrac{1}{2}].$$

The values taken by these spin characters are given by the following theorem (**13**).

VII. $[\lambda_1+\frac{1}{2}, \lambda_2+\frac{1}{2},..., \lambda_\nu+\frac{1}{2}] = [(\frac{1}{2})^\nu][\{\lambda\}+\sum(-1)^{\frac{1}{2}(p+r)}g_{\epsilon\eta\lambda}\{\eta\}]$ *summed for all S-functions* $\epsilon$ *which appear in* (11.9; 5).

Clearly the right-hand side of this equation represents a linear function of spin characters with positive or negative integral coefficients. Denoting this by $\zeta^{(\lambda)}$ we show that

$$\frac{1}{2h}\int \zeta^{(\lambda)^2}\,dT = 1,$$

which proves it to be a simple spin character.

Firstly, for $n = 2\nu$, let $\phi_1, \phi_2, \ldots, \phi_\nu$ be the parametric angles of a given rotation $S$. Let $U_1$ be an orthogonal matrix of negative determinant of degree $2\nu+1$, and with characteristic roots $-1$, $e^{\pm i\phi_1}, \ldots, e^{\pm i\phi_\nu}$. Let

$$h' = \int\limits_0^{2\pi} \cdots \int\limits_0^{2\pi} \prod (1+\cos\phi_r)\Delta(\cos\phi_r)\, d\phi_1 \ldots d\phi_\nu.$$

Then from Theorem V, § 11.9,

$$[\lambda]_1 = \{\lambda\} + \sum (-1)^{\frac{1}{2}(p+r)} g_{\epsilon\eta\lambda}\{\eta\}$$

is, for matrices of negative determinant, a simple character of the orthogonal group of degree $2\nu+1$, so that

$$\int\limits_0^{2\pi} \cdots \int\limits_0^{2\pi} [\lambda]_1^2 \prod (1+\cos\phi_r)\Delta(\cos\phi_r)\, d\phi_1 \ldots d\phi_\nu = h'.$$

Hence

$$\frac{1}{h} \int \zeta^{(\lambda)^2}\, dS = \frac{1}{h} \int\limits_0^{2\pi} \cdots \int\limits_0^{2\pi} [\lambda]_1^2 \prod 2(1+\cos\phi_r)\Delta(\cos\phi_r)\, d\phi_1 \ldots d\phi_\nu$$

$$= 2^\nu h'/h.$$

From the special case $[\lambda]_1 = 1$ it is clear that $2^\nu h'/h = 1$, and the theorem is proved for this case.

The case $n = 2\nu$ is proved in a similar manner. Let the characteristic roots of an element $S$ of positive determinant be

$$1,\ e^{\pm i\phi_1},\ \ldots,\ e^{\pm i\phi_\nu}.$$

Let $U$ be an orthogonal matrix of negative determinant of degree $2\nu+2$ with characteristic roots

$$\pm 1,\ e^{\pm i\phi_1},\ \ldots,\ e^{\pm i\phi_\nu}.$$

Let

$$h = \int\limits_0^{2\pi} \cdots \int\limits_0^{2\pi} \prod (1-\cos\phi_r)\Delta(\cos\phi_r)\, d\phi_1 \ldots d\phi_\nu,$$

$$h' = \int\limits_0^{2\pi} \cdots \int\limits_0^{2\pi} \prod (1-\cos^2\phi_r)\Delta(\cos\phi_r)\, d\phi_1 \ldots d\phi_\nu.$$

Now the values of the simple character of the orthogonal group of degree $2\nu+2$, corresponding to the partition $(\lambda)$, for elements of negative determinant, are given by

$$[\lambda]_1 = \{\lambda\}'' + \sum (-1)^{\frac{1}{2}p} g_{\gamma\eta\lambda}\{\eta\}'',$$

$\{\lambda\}''$ being an $S$-function of the quantities $\pm 1$, $e^{\pm i\phi_1},\ldots, e^{\pm i\phi_\nu}$, this expression being the coefficient of $\{x;\lambda\}$ in the expression

$$\frac{\prod(1-x_r^2)\prod(1-x_r x_s)}{\prod f(x_r)}. \tag{11.11;5}$$

By taking the factor $(1+x_r)$ from the expression $f(x_r)$, we see clearly that if $\{\lambda\}$ is an $S$-function of the characteristic roots of $S$, i.e. of

$$1,\ e^{\pm i\phi_1},\ \ldots,\ e^{\pm i\phi_\nu},$$

then
$$[\lambda]_1 = \{\lambda\}+\sum(-1)^{\frac{1}{2}(p+r)}g_{\epsilon\eta\lambda}\{\eta\}.$$

The proof now follows that for the case $n = 2\nu$. Since

$$\int_0^{2\pi}\cdots\int_0^{2\pi}[\lambda]_1^2\prod(1-\cos^2\phi_r)\Delta(\cos\phi_r)\,d\phi_1\ldots d\phi_\nu = h',$$

therefore

$$\frac{1}{h}\int_0^{2\pi}\cdots\int_0^{2\pi}[\lambda]_1^2\prod 2(1+\cos\phi_r)\prod(1-\cos\phi_r)\Delta(\cos\phi_r)\,d\phi_1\ldots d\phi_\nu = 2^\nu h'/h,$$

and $\zeta^{(\lambda)}$ is a simple spin character.

Again, for the case $n = 2\nu+1$, if we remove the factor $(1-x_r^2)$ from $f(x_r)$ in (11.11; 5), we see clearly that if $\{\lambda\}'$ denotes an $S$-function of the *variable* characteristic roots

$$e^{\pm i\phi_1},\ e^{\pm i\phi_2},\ \ldots,\ e^{\pm i\phi_\nu},$$

then
$$[\lambda]_1 = \{\lambda\}'+\sum(-1)^{\frac{1}{2}p}g_{\alpha\eta\lambda}\{\eta\}'.$$

We obtain thus the following theorem.

VIII. *If $n = 2\nu+1$ and $\{\lambda\}'$ denotes an S-function of*

$$e^{\pm i\phi_1},\ \ldots,\ e^{\pm i\phi_\nu},$$

*then the values of the spin characters are given by*

$$[\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2},\ldots,\lambda_\nu+\tfrac{1}{2}] = [(\tfrac{1}{2})^\nu][\{\lambda\}'+\sum(-1)^{\frac{1}{2}p}g_{\alpha\eta\lambda}\{\eta\}']$$

*summed for all S-functions $\{\alpha\}$ which appear in* (11.9; 1).

For the case $n = 2\nu$, by comparing Theorem VII with Theorems V and IX in § 11.9, we see that

$$[\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2},\ldots,\lambda_\nu+\tfrac{1}{2}] = \prod(2\cos\tfrac{1}{2}\phi_r)\frac{|C_t^{(\lambda_s+\nu-s+\frac{1}{2})}|}{|C_t^{(\nu-s+\frac{1}{2})}|},$$

where
$$C_i^{(j)} = 2\cos j\phi_i.$$

Now

$$\begin{aligned}\prod (2\cos\tfrac12\phi_r)\,|C_t^{(\nu-s)}| &\\ &= |2\cos\tfrac12\phi_t\,2\cos(\nu-s)\phi_t \,\vdots\, 2\cos\tfrac12\phi_t| \\ &= |2\cos(\nu-s+\tfrac12)\phi_t+2\cos(\nu-s-\tfrac12)\phi_t \,\vdots\, 2\cos\tfrac12\phi_t| \\ &= |2\cos(\nu-s+\tfrac12)\phi_t| = |C_t^{(\nu-s+\frac12)}|.\end{aligned}$$

In two of the determinants shown $s$ runs from 1 to $(\nu-1)$ before the dotted line, the case $s=\nu$ being given separately. Hence

IX. *For* $n=2\nu$,

$$[\lambda_1+\tfrac12,\lambda_2+\tfrac12,\ldots,\lambda_\nu+\tfrac12] = \frac{|C_t^{(\lambda_s+\nu-s+\frac12)}|}{|C_t^{(\nu-s)}|},$$

*where*

$$C_i^{(j)} = \begin{cases} 2\cos j\phi_i & (j\neq 0),\\ 1 & (j=0).\end{cases}$$

For $n=2\nu+1$, we compare Theorem VIII with Theorems III and VIII, § 11.9. We obtain

$$[\lambda_1+\tfrac12,\ldots,\lambda_\nu+\tfrac12] = \prod (2\cos\tfrac12\phi_r)\cdot\frac{|S_t^{(\lambda_s+\nu-s+1)}|}{|S_t^{(\nu-s+1)}|}$$

where

$$S_i^{(j)} = 2\sin j\phi_i.$$

Now

$$\begin{aligned}\prod (2\cos\tfrac12\phi_r)\,|S_t^{(\nu-s+\frac12)}| &= |2\cos\tfrac12\phi_t\,2\sin(\nu-s+\tfrac12)\phi_t| \\ &= |2\sin(\nu-s+1)\phi_t-2\sin(\nu-s)\phi_t| \\ &= |2\sin(\nu-s+1)\phi_t| = |S_t^{(\nu-s+1)}|.\end{aligned}$$

Hence

X. *For* $n=2\nu+1$,

$$[\lambda_1+\tfrac12,\lambda_2+\tfrac12,\ldots,\lambda_\nu+\tfrac12] = \frac{|S_t^{(\lambda_s+\nu-s+1)}|}{|S_t^{(\nu-s+\frac12)}|},$$

*where*

$$S_i^{(j)} = 2\sin j\phi_i.$$

These two theorems are clearly consistent with the corresponding results for true representations, and this fact provides another justification for the nomenclature for spin characters.

For the rotation group, for the case $n=2\nu$, clearly the conjugate spin characters into which a spin character of the orthogonal group separates, are obtained by replacing $\Delta$ by $\Delta_1$ and $\Delta_2$ respectively.

The *difference spin character* is thus obtained from the spin character by replacing $\prod (2\cos\frac{1}{2}\phi_r)$ by $\prod (2i\sin\frac{1}{2}\phi_r)$. It can be obtained as a quotient of determinants by dividing the determinant

$$|C_t^{(\nu-s+\frac{1}{2})}|$$

by $\prod (2\sin\frac{1}{2}\phi_r)$. We thus obtain

XI. *The difference spin characters of the rotation group are given by*

$$[\lambda_1+\tfrac{1}{2},\lambda_2+\tfrac{1}{2},\ldots,\lambda_\nu+\tfrac{1}{2}]' = \prod (2i\sin\phi_r)[\{\lambda\}-\sum(-1)^{\frac{1}{2}(p+r)}g_{\epsilon\eta\lambda}\{\eta\}]$$
$$= \frac{(i)^\nu|C_t^{(\lambda_s+\nu-s+\frac{1}{2})}|}{|S_t^{(\nu-s)}|},$$

*where*
$$C_k^{(j)} = 2\cos j\phi_k, \qquad S_k^{(j)} = 2\sin j\phi_k \qquad (j \neq 0),$$
$$C_k^{(0)} = 1, \qquad S_k^{(0)} = \tfrac{1}{2}\cot\phi_k.$$

## 11.12. Complex orthogonal matrices and groups of matrices with a quadratic invariant

The matrix representations of the real orthogonal group which have been obtained in the last two sections are algebraic. Hence if the real variables are replaced by complex variables, the results will be not at all affected.

I. *The irreducible matrix representations, both true and spin, of the group of real orthogonal matrices, give also irreducible matrix representations of the group of complex orthogonal matrices.*

Suppose that $A$ is the matrix of the linear transformation

$$Y = AX,$$

where $X$ and $Y$ are column vectors. If $A$ is orthogonal it leaves invariant the form
$$\sum x_r^2.$$

A more general group of matrices is that which leaves invariant the quadratic form

$$Q = \sum b_{pq}x_p x_q = X^tBX \quad (b_{pq} = b_{qp}), \qquad (11.12;1)$$

where $B = [b_{st}]$ and $X^t$ is the transpose of $X$.

$B$ is clearly a symmetric matrix. Hence we may find an orthogonal matrix $C$ such that

$$C^{-1}BC = C^tBC = \text{diag}(\lambda_1,\lambda_2,\ldots,\lambda_n),$$

where $\lambda_1,\ldots,\lambda_n$ are the characteristic roots of $B$.

If $\lambda_1,\ldots,\lambda_n$ are all positive, we may put

$$F = \text{diag}(\lambda_1^{\frac{1}{2}},\lambda_2^{\frac{1}{2}},\ldots,\lambda_n^{\frac{1}{2}}),$$

so that
$$C^tBC = F^2 = F^tF.$$

Putting $$G = CF^{-1},$$
we obtain, clearly, $$GG^t = B^{-1}. \qquad (11.12;2)$$

Now if $A$ leaves $Q$ invariant we have
$$X^tBX = Y^tBY = X^tA^tBAX,$$
so that $$B = A^tBA. \qquad (11.12;3)$$

Provided that the characteristic roots of $B$ are all positive, we may substitute from (11.12; 2) in (11.12; 3) and obtain
$$(G^t)^{-1}G^{-1} = A^t(G^t)^{-1}G^{-1}A,$$
so that $$G^tA^t(G^t)^{-1}.\,G^{-1}AG = I.$$

The required condition is that $G^{-1}AG$ is orthogonal.

II. *If the characteristic roots of a real matrix $B$ are all positive, the group of real matrices which leave $X^tBX$ invariant may be obtained by transforming the elements of the group of real orthogonal matrices by a real constant matrix $G$ satisfying*
$$GG^t = B^{-1}.$$
*The characters of the group are the same as the characters of the orthogonal group.*

The same result holds if all the characteristic roots of $B$ are negative, for $B$ may clearly be replaced by $-B$.

If the matrix $B$ has positive and negative roots, the matrix $G$ may be obtained in the same way, but it will not be a real matrix.

Referring, however, to Theorem I, we see that the group of complex matrices, with the quadratic invariant $X^tBX$, is equivalent to the group of complex orthogonal matrices, and has the same algebraic representations as the group of real orthogonal matrices. This will give a set of representations of the group of real matrices with the quadratic invariant $X^tBX$, for this is a subgroup of the group of complex matrices.

Furthermore, since the reducibility of a representation is obviously unaffected if a real parameter is replaced by a purely imaginary parameter, the representations will also be irreducible.

III. *The simple characters, both true and spin, of the group of real orthogonal matrices give also the simple characters of the group of real matrices with the quadratic invariant*
$$X^tBX.$$

An example is the Lorentz group in space-time, which leaves invariant the form

$$x_1^2+x_2^2+x_3^2-x_4^2.$$

For the cases which do not reduce to the orthogonal group, there are topological differences in the group manifolds which are worth noticing.

Firstly, if the group manifold extends to infinity, so that it is unclosed, there exist transcendental representations besides the algebraic representations which we have obtained.

Thus for the group of all non-singular real matrices of order $n^2$ there is a representation in which a matrix $S$ of determinant $\Delta$ is represented by

$$\Delta^\alpha\begin{bmatrix}1, & \beta\log\Delta \\ 0, & 1\end{bmatrix}.$$

For the group of matrices of order $2^2$ with the quadratic invariant $x_1^2-x_2^2$ there is a representation in which the matrix

$$\begin{bmatrix}\cosh\phi, & \sinh\phi \\ \sinh\phi, & \cosh\phi\end{bmatrix}$$

is represented by

$$\begin{bmatrix}\cosh k\phi, & \sinh k\phi \\ \sinh k\phi, & \cosh k\phi\end{bmatrix}.$$

This is transcendental if $k$ is not integral.

We shall not discuss the transcendental representations here, but merely mention them.

Secondly, if the characteristic roots of $B$ have not all the same sign, instead of the group manifold separating into two unconnected portions as in the case of the orthogonal group, it separates into four.

Clearly the group is equivalent to a group in which the quadratic invariant is

$$x_1^2+x_2^2+\ldots+x_r^2-x_{r+1}^2-\ldots-x_n^2.$$

We shall refer to the first $r$ variables as spatial variables, and the last $(n-r)$ as temporal variables in analogy with the Lorentz group.

The group manifold then divides into four unconnected portions which depend upon the sign of the determinant of the minor corresponding to the spatial coordinates, and upon the sign of the determinant of the minor corresponding to the temporal coordinates. We shall call these four sets $G^{++}$, $G^{+-}$, $G^{-+}$, $G^{--}$, the first sign indicating the sign of the spatial determinant, and the second the temporal.

The subgroup of positive determinant consists of the sets $G^{++}+G^{--}$. This corresponds to the rotation group except that the elements of $G^{--}$ cannot be generated by infinitesimal transformations.

The distinction between $G^{++}$ and $G^{--}$ leads to nothing new so far as the algebraic representations are concerned.

Corresponding to a rotation between a spatial and a temporal variable, there will be a matrix of the form

$$\begin{bmatrix} \cosh\phi, & \sinh\phi \\ \sinh\phi, & \cosh\phi \end{bmatrix}.$$

Every algebraic representation of this matrix gives also a representation of the matrix

$$\begin{bmatrix} -\cosh\phi, & -\sinh\phi \\ -\sinh\phi, & -\cosh\phi \end{bmatrix}.$$

The extra generator here involved extends $G^{++}$ to $G^{--}$, and $G^{+-}$ to $G^{-+}$ without affecting the representations.

There exist, however, differences as regards the non-algebraic representations.

It should be noticed, for example, that the obvious representation of the subgroup of positive determinant in which elements of $G^{++}$ are represented by $+1$, and elements of $G^{--}$ by $-1$, is not an algebraic representation. That is, there does not exist a polynomial in the elements of a matrix $S$ which gives $+1$ for an element of $G^{++}$ and $-1$ for an element of $G^{--}$.

# APPENDIX

## *Tables of Characters of the Symmetric Groups*

### *Degree* 2

| Class | $1^2$ | 2 |
|---|---|---|
| Order | 1 | 1 |
| [2] | 1 | 1 |
| $[1^2]$ | 1 | −1 |

### *Degree* 3

| Class | $1^3$ | 12 | 3 |
|---|---|---|---|
| Order | 1 | 3 | 2 |
| [3] | 1 | 1 | 1 |
| †[2 1] | 2 | 0 | −1 |
| $[1^3]$ | 1 | −1 | 1 |

### *Degree* 4

| Class | $1^4$ | $1^2\,2$ | 1 3 | 4 | $2^2$ |
|---|---|---|---|---|---|
| Order | 1 | 6 | 8 | 6 | 3 |
| [4] | 1 | 1 | 1 | 1 | 1 |
| [3 1] | 3 | 1 | 0 | −1 | −1 |
| †$[2^2]$ | 2 | 0 | −1 | 0 | 2 |
| $[2\,1^2]$ | 3 | −1 | 0 | 1 | −1 |
| $[1^4]$ | 1 | −1 | 1 | −1 | 1 |

### *Degree* 5

| Class | $1^5$ | $1^3\,2$ | $1^2\,3$ | 1 4 | $1\,2^2$ | 2 3 | 5 |
|---|---|---|---|---|---|---|---|
| Order | 1 | 10 | 20 | 30 | 15 | 20 | 24 |
| [5] | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| [4 1] | 4 | 2 | 1 | 0 | 0 | −1 | −1 |
| [3 2] | 5 | 1 | −1 | −1 | 1 | 1 | 0 |
| †$[3\,1^2]$ | 6 | 0 | 0 | 0 | −2 | 0 | 1 |
| $[2^2\,1]$ | 5 | −1 | −1 | 1 | 1 | −1 | 0 |
| $[2\,1^3]$ | 4 | −2 | 1 | 0 | 0 | 1 | −1 |
| $[1^5]$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

† Denotes a self-associated partition and character.

## *Degree* 6

| Class | $1^6$ | $1^4\,2$ | $1^3\,3$ | $1^2\,4$ | $1^2\,2^2$ | $1\,2\,3$ | $1\,5$ | $6$ | $2\,4$ | $2^3$ | $3^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 15 | 40 | 90 | 45 | 120 | 144 | 120 | 90 | 15 | 40 |
| $[6]$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[5\,1]$ | 5 | 3 | 2 | 1 | 1 | 0 | 0 | −1 | −1 | −1 | −1 |
| $[4\,2]$ | 9 | 3 | 0 | −1 | 1 | 0 | −1 | 0 | 1 | 3 | 0 |
| $[4\,1^2]$ | 10 | 2 | 1 | 0 | −2 | −1 | 0 | 1 | 0 | −2 | 1 |
| $[3^2]$ | 5 | 1 | −1 | −1 | 1 | 1 | 0 | 0 | −1 | −3 | 2 |
| †$[3\,2\,1]$ | 16 | 0 | −2 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −2 |
| $[2^3]$ | 5 | −1 | −1 | 1 | 1 | −1 | 0 | 0 | −1 | 3 | 2 |
| $[3\,1^3]$ | 10 | −2 | 1 | 0 | −2 | 1 | 0 | −1 | 0 | 2 | 1 |
| $[2^2\,1^2]$ | 9 | −3 | 0 | 1 | 1 | 0 | −1 | 0 | 1 | −3 | 0 |
| $[2\,1^4]$ | 5 | −3 | 2 | −1 | 1 | 0 | 0 | 1 | −1 | 1 | −1 |
| $[1^6]$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

## *Degree* 7

| Class | $1^7$ | $1^5\,2$ | $1^4\,3$ | $1^3\,4$ | $1^3\,2^2$ | $1^2\,2\,3$ | $1^2\,5$ | $1\,6$ | $1\,2\,4$ | $1\,2^3$ | $1\,3^2$ | $2\,5$ | $2^2\,3$ | $3\,4$ | $7$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 21 | 70 | 210 | 105 | 420 | 504 | 840 | 630 | 105 | 280 | 504 | 210 | 420 | 720 |
| $[7]$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[6\,1]$ | 6 | 4 | 3 | 2 | 2 | 1 | 1 | 0 | 0 | 0 | 0 | −1 | −1 | −1 | −1 |
| $[5\,2]$ | 14 | 6 | 2 | 0 | 2 | 0 | −1 | −1 | 0 | 2 | −1 | 1 | 2 | 0 | 0 |
| $[5\,1^2]$ | 15 | 5 | 3 | 1 | −1 | −1 | 0 | 0 | −1 | −3 | 0 | 0 | −1 | 1 | 1 |
| $[4\,3]$ | 14 | 4 | −1 | −2 | 2 | 1 | −1 | 0 | 0 | 0 | 2 | −1 | −1 | 1 | 0 |
| $[4\,2\,1]$ | 35 | 5 | −1 | −1 | −1 | −1 | 0 | 1 | 1 | 1 | −1 | 0 | −1 | −1 | 0 |
| $[3^2\,1]$ | 21 | 1 | −3 | −1 | 1 | 1 | 1 | 0 | −1 | −3 | 0 | 1 | 1 | −1 | 0 |
| †$[4\,1^3]$ | 20 | 0 | 2 | 0 | −4 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 2 | 0 | −1 |
| $[3\,2^2]$ | 21 | −1 | −3 | 1 | 1 | −1 | 1 | 0 | −1 | 3 | 0 | −1 | 1 | 1 | 0 |
| $[3\,2\,1^2]$ | 35 | −5 | −1 | 1 | −1 | 1 | 0 | −1 | 1 | −1 | −1 | 0 | −1 | 1 | 0 |
| $[2^3\,1]$ | 14 | −4 | −1 | 2 | 2 | −1 | −1 | 0 | 0 | 0 | 2 | 1 | −1 | −1 | 0 |
| $[3\,1^4]$ | 15 | −5 | 3 | −1 | −1 | 1 | 0 | 0 | −1 | 3 | 0 | 0 | −1 | −1 | 1 |
| $[2^2\,1^3]$ | 14 | −6 | 2 | 0 | 2 | 0 | −1 | 1 | 0 | −2 | −1 | −1 | 2 | 0 | 0 |
| $[2\,1^5]$ | 6 | −4 | 3 | −2 | 2 | −1 | 1 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | −1 |
| $[1^7]$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

† Denotes a self-associated partition and character.

## *Degree* 8

| Class | $1^8$ | $1^6\,2$ | $1^5\,3$ | $1^4\,4$ | $1^4\,2^2$ | $1^3\,2\,3$ | $1^3\,5$ | $1^2\,6$ | $1^2\,2\,4$ | $1^2\,2^3$ | $1^2\,3^2$ | $1\,2\,5$ | $1\,2^2\,3$ | $1\,3\,4$ | $1\,7$ | $8$ | $4^2$ | $2^2\,4$ | $2\,6$ | $2\,3^2$ | $3\,5$ | $2^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 28 | 112 | 420 | 210 | 1120 | 1344 | 3360 | 2520 | 420 | 1120 | 4032 | 1680 | 3360 | 5760 | 5040 | 1260 | 1260 | 3360 | 1120 | 2688 | 105 |
| $[8]$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[7\,1]$ | 7 | 5 | 4 | 3 | 3 | 2 | 2 | 1 | 1 | 1 | 1 | 0 | 0 | 0 | 0 | −1 | −1 | −1 | −1 | −1 | −1 | −1 |
| $[6\,2]$ | 20 | 10 | 5 | 2 | 4 | 1 | 0 | −1 | 0 | 2 | −1 | 0 | 1 | −1 | −1 | 0 | 0 | 2 | 1 | 1 | 0 | 4 |
| $[6\,1^2]$ | 21 | 9 | 6 | 3 | 1 | 0 | 1 | 0 | −1 | −3 | 0 | −1 | −2 | 0 | 0 | 1 | 1 | −1 | 0 | 0 | 1 | −3 |
| $[5\,3]$ | 28 | 10 | 1 | −2 | 4 | 1 | −2 | −1 | 0 | 2 | 1 | 0 | 1 | 1 | 0 | 0 | 0 | −2 | −1 | 1 | 1 | −4 |
| $[5\,2\,1]$ | 64 | 16 | 4 | 0 | 0 | −2 | −1 | 0 | 0 | 0 | −2 | 1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | −2 | −1 | 0 |
| $[5\,1^3]$ | 35 | 5 | 5 | 1 | −5 | −1 | 0 | 0 | −1 | −3 | 2 | 0 | 1 | 1 | 0 | −1 | −1 | 1 | 0 | 2 | 0 | 3 |
| $[4^2]$ | 14 | 4 | −1 | −2 | 2 | 1 | −1 | 0 | 0 | 0 | 2 | −1 | −1 | 1 | 0 | 0 | 2 | 2 | 0 | −2 | −1 | 6 |
| $[4\,3\,1]$ | 70 | 10 | −5 | −4 | 2 | 1 | 0 | 1 | 0 | −2 | 1 | 0 | −1 | −1 | 0 | 0 | −2 | 0 | 1 | 1 | 0 | −2 |
| $[4\,2^2]$ | 56 | 4 | −4 | 0 | 0 | −2 | 1 | 1 | 0 | 4 | −1 | −1 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | 1 | 8 |
| †$[4\,2\,1^2]$ | 90 | 0 | 0 | 0 | −6 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | −1 | 0 | 2 | 0 | 0 | 0 | 0 | −6 |
| †$[3^2\,2]$ | 42 | 0 | −6 | 0 | 2 | 0 | 2 | 0 | −2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | −1 | −6 |
| $[3^2\,1^2]$ | 56 | −4 | −4 | 0 | 0 | 2 | 1 | −1 | 0 | −4 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | 1 | 8 |
| $[3\,2^2\,1]$ | 70 | −10 | −5 | 4 | 2 | −1 | 0 | −1 | 0 | 2 | 1 | 0 | −1 | 1 | 0 | 0 | −2 | 0 | 1 | −1 | 0 | −2 |
| $[2^4]$ | 14 | −4 | −1 | 2 | 2 | −1 | −1 | 0 | 0 | 0 | 2 | 1 | −1 | −1 | 0 | 0 | 2 | −2 | 0 | 2 | −1 | 6 |
| $[4\,1^4]$ | 35 | −5 | 5 | −1 | −5 | 1 | 0 | 0 | −1 | 3 | 2 | 0 | 1 | −1 | 0 | 1 | −1 | −1 | 0 | −2 | 0 | 3 |
| $[3\,2\,1^3]$ | 64 | −16 | 4 | 0 | 0 | 2 | −1 | 0 | 0 | 0 | −2 | −1 | 0 | 0 | 1 | 0 | 0 | 0 | 0 | 2 | −1 | 0 |
| $[2^3\,1^2]$ | 28 | −10 | 1 | 2 | 4 | −1 | −2 | 1 | 0 | −2 | 1 | 0 | 1 | −1 | 0 | 0 | 0 | 2 | −1 | −1 | 1 | −4 |
| $[3\,1^5]$ | 21 | −9 | 6 | −3 | 1 | 0 | 1 | 0 | −1 | 3 | 0 | 1 | −2 | 0 | 0 | −1 | 1 | 1 | 0 | 0 | 1 | −3 |
| $[2^2\,1^4]$ | 20 | −10 | 5 | −2 | 4 | −1 | 0 | 1 | 0 | −2 | −1 | 0 | 1 | 1 | −1 | 0 | 0 | −2 | 1 | −1 | 0 | 4 |
| $[2\,1^6]$ | 7 | −5 | 4 | −3 | 3 | −2 | 2 | −1 | 1 | −1 | 1 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | −1 | 1 | −1 | −1 |
| $[1^8]$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | 1 |

† Denotes a self-associated partition and character.

## *Degree* 9

| Class | $1^9$ | $1^7 2$ | $1^6 3$ | $1^5 4$ | $1^5 2^2$ | $1^4 2 3$ | $1^4 5$ | $1^3 6$ | $1^3 2 4$ | $1^3 2^3$ | $1^3 3^2$ | $1^2 2 5$ | $1^2 2^2 3$ | $1^2 3 4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 36 | 168 | 756 | 378 | 2520 | 3024 | 10080 | 7560 | 1260 | 3360 | 18144 | 7560 | 15120 |
| $[9]$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $[81]$ | 8 | 6 | 5 | 4 | 4 | 3 | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 1 |
| $[72]$ | 27 | 15 | 9 | 5 | 7 | 3 | 2 | 0 | 1 | 3 | 0 | 0 | 1 | −1 |
| $[71^2]$ | 28 | 14 | 10 | 6 | 4 | 2 | 3 | 1 | 0 | −2 | 1 | −1 | −2 | 0 |
| $[63]$ | 48 | 20 | 6 | 0 | 8 | 2 | −2 | −2 | 0 | 4 | 0 | 0 | 2 | 0 |
| $[621]$ | 105 | 35 | 15 | 5 | 5 | −1 | 0 | −1 | −1 | −1 | −3 | 0 | −1 | −1 |
| $[61^3]$ | 56 | 14 | 11 | 4 | −4 | −1 | 1 | 0 | −2 | −6 | 2 | −1 | −1 | 1 |
| $[54]$ | 42 | 14 | 0 | −4 | 6 | 2 | −3 | −1 | 0 | 2 | 3 | −1 | 0 | 2 |
| $[531]$ | 162 | 36 | 0 | −6 | 6 | 0 | −3 | 0 | 0 | 0 | 0 | 1 | 0 | 0 |
| $[52^2]$ | 120 | 20 | 0 | 0 | 0 | −4 | 0 | 1 | 0 | 4 | −3 | 0 | 0 | 0 |
| $[521^2]$ | 189 | 21 | 9 | 1 | −11 | −3 | −1 | 0 | 1 | −3 | 0 | 1 | 1 | 1 |
| $[4^2 1]$ | 84 | 14 | −6 | −6 | 4 | 2 | −1 | 1 | 0 | −2 | 3 | −1 | −2 | 0 |
| $[432]$ | 168 | 14 | −15 | −4 | 4 | −1 | 3 | 2 | −2 | 2 | 0 | −1 | 1 | −1 |
| $[431^2]$ | 216 | 6 | −9 | −4 | −4 | 3 | 1 | 0 | 2 | −6 | 0 | 1 | −1 | −1 |
| †$[51^4]$ | 70 | 0 | 10 | 0 | −10 | 0 | 0 | 0 | −2 | 0 | 4 | 0 | 2 | 0 |
| †$[3^3]$ | 42 | 0 | −6 | 0 | 2 | 0 | 2 | 0 | −2 | 0 | 0 | 0 | 2 | 0 |
| $[42^2 1]$ | 216 | −6 | −9 | 4 | −4 | −3 | 1 | 0 | 2 | 6 | 0 | −1 | −1 | 1 |
| $[3^2 21]$ | 168 | −14 | −15 | 4 | 4 | 1 | 3 | −2 | −2 | −2 | 0 | 1 | 1 | 1 |
| $[32^3]$ | 84 | −14 | −6 | 6 | 4 | −2 | −1 | −1 | 0 | 2 | 3 | 1 | −2 | 0 |
| $[421^3]$ | 189 | −21 | 9 | −1 | −11 | 3 | −1 | 0 | 1 | 3 | 0 | −1 | 1 | −1 |
| $[3^2 1^3]$ | 120 | −20 | 0 | 0 | 0 | 4 | 0 | −1 | 0 | −4 | −3 | 0 | 0 | 0 |
| $[32^2 1^2]$ | 162 | −36 | 0 | 6 | 6 | 0 | −3 | 0 | 0 | 0 | 0 | −1 | 0 | 0 |
| $[2^4 1]$ | 42 | −14 | 0 | 4 | 6 | −2 | −3 | 1 | 0 | −2 | 3 | 1 | 0 | −2 |
| $[41^5]$ | 56 | −14 | 11 | −4 | −4 | 1 | 1 | 0 | −2 | 6 | 2 | 1 | −1 | −1 |
| $[321^4]$ | 105 | −35 | 15 | −5 | 5 | 1 | 0 | 1 | −1 | 1 | −3 | 0 | −1 | 1 |
| $[2^3 1^3]$ | 48 | −20 | 6 | 0 | 8 | −2 | −2 | 2 | 0 | −4 | 0 | 0 | 2 | 0 |
| $[31^6]$ | 28 | −14 | 10 | −6 | 4 | −2 | 3 | −1 | 0 | 2 | 1 | 1 | −2 | 0 |
| $[2^2 1^5]$ | 27 | −15 | 9 | −5 | 7 | −3 | 2 | 0 | 1 | −3 | 0 | 0 | 1 | 1 |
| $[21^7]$ | 8 | −6 | 5 | −4 | 4 | −3 | 3 | −2 | 2 | −2 | 2 | −1 | 1 | −1 |
| $[1^9]$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |

† Denotes a self-associated partition and character.

## *Degree* 9 (*cont.*)

| $1^2 7$ | 18 | $14^2$ | $12^2 4$ | 126 | $123^2$ | 135 | $12^4$ | 45 | $3^3$ | 36 | 234 | 27 | $2^2 5$ | $2^3 3$ | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 25920 | 45360 | 11340 | 11340 | 30240 | 10080 | 24192 | 945 | 18144 | 2240 | 20160 | 15120 | 25920 | 9072 | 2520 | 40320 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 |
| −1 | −1 | −1 | 1 | 0 | 0 | −1 | 3 | 0 | 0 | 0 | 1 | 1 | 2 | 3 | 0 |
| 0 | 0 | 0 | −2 | −1 | −1 | 0 | −4 | 1 | 1 | 1 | 0 | 0 | −1 | −2 | 1 |
| −1 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | 0 | 3 | 1 | 0 | −1 | −2 | −2 | 0 |
| 0 | 1 | 1 | 1 | 1 | −1 | 0 | 1 | 0 | −3 | −1 | −1 | 0 | 0 | −1 | 0 |
| 0 | 0 | 0 | 0 | 0 | 2 | 1 | 0 | −1 | 2 | 0 | 1 | 0 | 1 | 3 | −1 |
| 0 | 0 | 2 | 0 | −1 | −1 | 0 | 2 | 1 | −3 | −1 | 0 | 0 | 1 | 2 | 0 |
| 1 | 0 | −2 | −2 | 0 | 0 | 0 | −6 | −1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 |
| 1 | 0 | 0 | 0 | −1 | −1 | 0 | 8 | 0 | 3 | 1 | 0 | −1 | 0 | 4 | 0 |
| 0 | −1 | 1 | 1 | 0 | 0 | −1 | −3 | 1 | 0 | 0 | 1 | 0 | −1 | −3 | 0 |
| 0 | 0 | 0 | 2 | 1 | −1 | −1 | 4 | −1 | 3 | 1 | 0 | 0 | −1 | −2 | 0 |
| 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 1 | −3 | −1 | 1 | 0 | −1 | −1 | 0 |
| −1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 1 | 0 | 0 | −1 | −1 | 1 | 3 | 0 |
| 0 | 0 | −2 | 0 | 0 | 0 | 0 | 6 | 0 | −2 | 0 | −2 | 0 | 0 | 0 | 1 |
| 0 | 0 | 2 | 0 | 0 | 0 | −1 | −6 | 0 | 6 | 0 | −2 | 0 | 2 | 0 | 0 |
| −1 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | −1 | 0 | 0 | −1 | 1 | 1 | −3 | 0 |
| 0 | 0 | 0 | 0 | 0 | −2 | 0 | 0 | −1 | −3 | 1 | 1 | 0 | −1 | 1 | 0 |
| 0 | 0 | 0 | −2 | 1 | 1 | −1 | 4 | 1 | 3 | −1 | 0 | 0 | −1 | 2 | 0 |
| 0 | 1 | 1 | −1 | 0 | 0 | −1 | −3 | −1 | 0 | 0 | 1 | 0 | −1 | 3 | 0 |
| 1 | 0 | 0 | 0 | −1 | 1 | 0 | 8 | 0 | 3 | −1 | 0 | 1 | 0 | −4 | 0 |
| 1 | 0 | −2 | 2 | 0 | 0 | 0 | −6 | 1 | 0 | 0 | 0 | −1 | 1 | 0 | 0 |
| 0 | 0 | 2 | 0 | −1 | 1 | 0 | 2 | −1 | −3 | 1 | 0 | 0 | 1 | −2 | 0 |
| 0 | 0 | 0 | 0 | 0 | −2 | 1 | 0 | 1 | 2 | 0 | 1 | 0 | 1 | −3 | −1 |
| 0 | −1 | 1 | −1 | 1 | 1 | 0 | 1 | 0 | −3 | 1 | −1 | 0 | 0 | 1 | 0 |
| −1 | 0 | 0 | 0 | 0 | −2 | 1 | 0 | 0 | 3 | −1 | 0 | 1 | −2 | 2 | 0 |
| 0 | 0 | 0 | 2 | −1 | 1 | 0 | −4 | −1 | 1 | −1 | 0 | 0 | −1 | 2 | 1 |
| −1 | 1 | −1 | −1 | 0 | 0 | −1 | 3 | 0 | 0 | 0 | 1 | −1 | 2 | −3 | 0 |
| 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| 1 | −1 | 1 | −1 | 1 | −1 | 1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

## *Degree* 10

| Class | $1^{10}$ | $1^8 2$ | $1^7 3$ | $1^6 4$ | $1^6 2^2$ | $1^5 23$ | $1^5 5$ | $1^4 6$ | $1^4 24$ | $1^4 2^3$ | $1^4 3^2$ | $1^3 25$ | $1^3 2^2 3$ | $1^3 34$ | $1^3 7$ | $1^2 8$ | $1^2 4^2$ | $1^2 2^2 4$ | $1^2 26$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 45 | 240 | 1260 | 630 | 5040 | 6048 | 25200 | 18900 | 3150 | 8400 | 60480 | 25200 | 50400 | 86400 | 226800 | 56700 | 56700 | 151200 |
| [10] | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| [91] | 9 | 7 | 6 | 5 | 5 | 4 | 4 | 3 | 3 | 3 | 3 | 2 | 2 | 2 | 2 | 1 | 1 | 1 | 1 |
| [82] | 35 | 21 | 14 | 9 | 11 | 6 | 5 | 2 | 3 | 5 | 2 | 1 | 2 | 0 | 0 | −1 | −1 | 1 | 0 |
| $[81^2]$ | 36 | 20 | 15 | 10 | 8 | 5 | 6 | 3 | 2 | 0 | 3 | 0 | −1 | 1 | 1 | 0 | 0 | −2 | −1 |
| [73] | 75 | 35 | 15 | 5 | 15 | 5 | 0 | −2 | 1 | 7 | 0 | 0 | 3 | −1 | −2 | −1 | −1 | 1 | 0 |
| [721] | 160 | 64 | 34 | 16 | 16 | 4 | 5 | 0 | 0 | 0 | −2 | −1 | −2 | −2 | −1 | 0 | 0 | 0 | 0 |
| $[71^3]$ | 84 | 28 | 21 | 10 | 0 | 1 | 4 | 1 | −2 | −8 | 3 | −2 | −3 | 1 | 0 | 0 | 0 | −2 | −1 |
| [64] | 90 | 34 | 6 | −4 | 14 | 4 | −5 | −3 | 0 | 6 | 3 | −1 | 2 | 2 | −1 | 0 | 2 | 0 | −1 |
| [631] | 315 | 91 | 21 | −1 | 19 | 1 | −5 | −3 | −1 | 3 | −3 | 1 | 1 | −1 | 0 | 1 | −1 | −1 | 1 |
| $[62^2]$ | 225 | 55 | 15 | 5 | 5 | −5 | 0 | 0 | −1 | 3 | −6 | 0 | −1 | −1 | 1 | 1 | 1 | 1 | 0 |
| $[621^2]$ | 350 | 70 | 35 | 10 | −10 | −5 | 0 | −1 | −2 | −10 | −1 | 0 | −1 | 1 | 0 | 0 | 2 | 2 | 1 |
| $[61^4]$ | 126 | 14 | 21 | 4 | −14 | −1 | 1 | 0 | −4 | −6 | 6 | −1 | 1 | 1 | 0 | 0 | −2 | 0 | 0 |
| $[5^2]$ | 42 | 14 | 0 | −4 | 6 | 2 | −3 | −1 | 0 | 2 | 3 | −1 | 0 | 2 | 0 | 0 | 2 | 0 | −1 |
| [541] | 288 | 64 | −6 | −16 | 16 | 4 | −7 | 0 | 0 | 0 | 6 | −1 | −2 | 2 | 1 | 0 | 0 | 0 | 0 |
| [532] | 450 | 70 | −15 | −10 | 10 | −5 | 0 | 3 | −2 | 6 | −3 | 0 | 1 | −1 | 2 | 0 | −2 | −2 | −1 |
| $[531^2]$ | 567 | 63 | 0 | −9 | −9 | 0 | −3 | 0 | 3 | −9 | 0 | 3 | 0 | 0 | 0 | −1 | −1 | −1 | 0 |
| $[52^2 1]$ | 525 | 35 | 0 | 5 | −15 | −10 | 0 | 1 | 3 | 7 | −3 | 0 | 0 | 2 | 0 | −1 | 1 | 1 | −1 |
| $[4^2 2]$ | 252 | 28 | −21 | −10 | 8 | 1 | 2 | 3 | −2 | 0 | 3 | −2 | −1 | −1 | 0 | 0 | 0 | 2 | 1 |
| $[4^2 1^2]$ | 300 | 20 | −15 | −10 | 0 | 5 | 0 | 1 | 2 | −8 | 3 | 0 | −3 | −1 | −1 | 0 | 0 | 2 | 1 |
| $[43^2]$ | 210 | 14 | −21 | −4 | 6 | −1 | 5 | 2 | −4 | 2 | 0 | −1 | 3 | −1 | 0 | 0 | 2 | 0 | 0 |
| †[4321] | 768 | 0 | −48 | 0 | 0 | 0 | 8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −2 | 0 | 0 | 0 | 0 |
| †$[521^3]$ | 448 | 0 | 28 | 0 | −32 | 0 | −2 | 0 | 0 | 0 | 4 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | 0 |
| $[3^3 1]$ | 210 | −14 | −21 | 4 | 6 | 1 | 5 | −2 | −4 | −2 | 0 | 1 | 3 | 1 | 0 | 0 | 2 | 0 | 0 |
| $[42^3]$ | 300 | −20 | −15 | 10 | 0 | −5 | 0 | −1 | 2 | 8 | 3 | 0 | −3 | 1 | −1 | 0 | 0 | −2 | 1 |
| $[3^2 2^2]$ | 252 | −28 | −21 | 10 | 8 | −1 | 2 | −3 | −2 | 0 | 3 | 2 | −1 | 1 | 0 | 0 | 0 | −2 | 1 |
| $[431^3]$ | 525 | −35 | 0 | −5 | −15 | 10 | 0 | −1 | 3 | −7 | −3 | 0 | 0 | −2 | 0 | 1 | 1 | −1 | −1 |
| $[42^2 1^2]$ | 567 | −63 | 0 | 9 | −9 | 0 | −3 | 0 | 3 | 9 | 0 | −3 | 0 | 0 | 0 | 1 | −1 | 1 | 0 |
| $[3^2 21^2]$ | 450 | −70 | −15 | 10 | 10 | 5 | 0 | −3 | −2 | −6 | −3 | 0 | 1 | 1 | 2 | 0 | −2 | 2 | −1 |
| $[32^3 1]$ | 288 | −64 | −6 | 16 | 16 | −4 | −7 | 0 | 0 | 0 | 6 | 1 | −2 | −2 | 1 | 0 | 0 | 0 | 0 |
| $[2^5]$ | 42 | −14 | 0 | 4 | 6 | −2 | −3 | 1 | 0 | −2 | 3 | 1 | 0 | −2 | 0 | 0 | 2 | 0 | −1 |
| $[51^5]$ | 126 | −14 | 21 | −4 | −14 | 1 | 1 | 0 | −4 | 6 | 6 | 1 | 1 | −1 | 0 | 0 | −2 | 0 | 0 |
| $[421^4]$ | 350 | −70 | 35 | −10 | −10 | 5 | 0 | 1 | −2 | 10 | −1 | 0 | −1 | −1 | 0 | 0 | 2 | −2 | 1 |
| $[3^2 1^4]$ | 225 | −55 | 15 | −5 | 5 | 5 | 0 | 0 | −1 | −3 | −6 | 0 | −1 | 1 | 1 | −1 | 1 | −1 | 0 |
| $[32^2 1^3]$ | 315 | −91 | 21 | 1 | 19 | −1 | −5 | 3 | −1 | −3 | −3 | −1 | 1 | 1 | 0 | −1 | −1 | 1 | 1 |
| $[2^4 1^2]$ | 90 | −34 | 6 | 4 | 14 | −4 | −5 | 3 | 0 | −6 | 3 | 1 | 2 | −2 | −1 | 0 | 2 | 0 | −1 |
| $[41^6]$ | 84 | −28 | 21 | −10 | 0 | −1 | 4 | −1 | −2 | 8 | 3 | 2 | −3 | −1 | 0 | 0 | 0 | 2 | −1 |
| $[321^5]$ | 160 | −64 | 34 | −16 | 16 | −4 | 5 | 0 | 0 | 0 | −2 | 1 | −2 | 2 | −1 | 0 | 0 | 0 | 0 |
| $[2^3 1^4]$ | 75 | −35 | 15 | −5 | 15 | −5 | 0 | 2 | 1 | −7 | 0 | 0 | 3 | 1 | −2 | 1 | −1 | −1 | 0 |
| $[31^7]$ | 36 | −20 | 15 | −10 | 8 | −5 | 6 | −3 | 2 | 0 | 3 | 0 | −1 | −1 | 1 | 0 | 0 | 2 | −1 |
| $[2^2 1^6]$ | 35 | −21 | 14 | −9 | 11 | −6 | 5 | −2 | 3 | −5 | 2 | −1 | 2 | 0 | 0 | 1 | −1 | −1 | 0 |
| $[21^8]$ | 9 | −7 | 6 | −5 | 5 | −4 | 4 | −3 | 3 | −3 | 3 | −2 | 2 | −2 | 2 | −1 | 1 | −1 | 1 |
| $[1^{10}]$ | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

† Denotes a self-associated partition and character.

## *Degree* 10 (*cont.*)

| $1^2 2 3^2$ | $1^2 3 5$ | $1^2 2^4$ | $1 4 5$ | $1 3^3$ | $1 3 6$ | $1 2 3 4$ | $1 2 7$ | $1 2^2 5$ | $1 2^3 3$ | $1 9$ | $10$ | $8 2$ | $6 2^2$ | $4 2^3$ | $2^5$ | $7 3$ | $5 3 2$ | $6 4$ | $4^2 2$ | $5^2$ | $4 3^2$ | $3^2 2^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 50400 | 120960 | 4725 | 181440 | 22400 | 201600 | 151200 | 259200 | 90720 | 25200 | 403200 | 362880 | 226800 | 75600 | 18900 | 945 | 72800 | 20960 | 51200 | 56700 | 72576 | 50400 | 25200 |
| 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| 1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 | −1 |
| 0 | −1 | 3 | −1 | −1 | −1 | 0 | 0 | 1 | 2 | −1 | 0 | 1 | 2 | 3 | 5 | 0 | 1 | 0 | 1 | 0 | 0 | 2 |
| −1 | 0 | −4 | 0 | 0 | 0 | −1 | −1 | −2 | −3 | 0 | 1 | 0 | −1 | −2 | −4 | 1 | 0 | 1 | 0 | 1 | 1 | −1 |
| 2 | 0 | 3 | 0 | 3 | 1 | 1 | 0 | 0 | 1 | 0 | 0 | −1 | −2 | −3 | −5 | 1 | 0 | 0 | −1 | 0 | 2 | 0 |
| −2 | −1 | 0 | 1 | −2 | 0 | 0 | 1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | 0 | 0 | 0 | −2 | −2 |
| 1 | 1 | −4 | 0 | 3 | 1 | 1 | 0 | 0 | 1 | 0 | −1 | 0 | 1 | 2 | 4 | 0 | 1 | −1 | 0 | −1 | 1 | 3 |
| 1 | 1 | 2 | 1 | 0 | 0 | 0 | −1 | −1 | 0 | 0 | 0 | 0 | 1 | 4 | 10 | −1 | −1 | 1 | 2 | 0 | −1 | −1 |
| 1 | 1 | −5 | −1 | 0 | 0 | −1 | 0 | −1 | −3 | 0 | 0 | 1 | 1 | −1 | −5 | 0 | 1 | −1 | −1 | 0 | −1 | 1 |
| −2 | 0 | 9 | 0 | 0 | 0 | −1 | −1 | 0 | 3 | 0 | 0 | −1 | 0 | 3 | 15 | 1 | 0 | 0 | −1 | 0 | 2 | 2 |
| 1 | 0 | −2 | 0 | −1 | −1 | 1 | 0 | 0 | −1 | −1 | 0 | 0 | −1 | −2 | −10 | 0 | 0 | 1 | 2 | 0 | 1 | −1 |
| 2 | 1 | 6 | −1 | 0 | 0 | −1 | 0 | 1 | 3 | 0 | 1 | 0 | 0 | 0 | 6 | 0 | −1 | 0 | −2 | 1 | −2 | −2 |
| −1 | 0 | 2 | 1 | −3 | −1 | 0 | 0 | 1 | 2 | 0 | 0 | 0 | −1 | −4 | −10 | 0 | 2 | −1 | −2 | 2 | −1 | 3 |
| −2 | −1 | 0 | −1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | −2 | 2 | −2 |
| 1 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | 3 | 0 | 0 | 0 | −1 | −2 | −10 | −1 | 0 | 1 | 2 | 0 | −1 | 1 |
| 0 | 0 | −9 | 1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −1 | 0 | 3 | 15 | 0 | 0 | 0 | −1 | 2 | 0 | 0 |
| −1 | 0 | 5 | 0 | 3 | 1 | 0 | 0 | 0 | −2 | 0 | 0 | 1 | 1 | −1 | −5 | 0 | 0 | −1 | −1 | 0 | −1 | −3 |
| 1 | −1 | 4 | 0 | 0 | 0 | 1 | 0 | −2 | −3 | 0 | 0 | 0 | −1 | 2 | 20 | 0 | 1 | −1 | 0 | 2 | −1 | −1 |
| −1 | 0 | 4 | 0 | 3 | 1 | −1 | −1 | 0 | 1 | 0 | 0 | 0 | 1 | −2 | −20 | −1 | 0 | 1 | 0 | 0 | −1 | 3 |
| 2 | −1 | −6 | 1 | 3 | −1 | −1 | 0 | 1 | −1 | 0 | 0 | 0 | 2 | 0 | −10 | 0 | −1 | 0 | −2 | 0 | 2 | 0 |
| 0 | 2 | 0 | 0 | −6 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −2 | 0 | 0 |
| 0 | −2 | 0 | 0 | −2 | 0 | 0 | 0 | −2 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | −2 | 0 | 4 |
| −2 | −1 | −6 | −1 | 3 | 1 | −1 | 0 | 1 | 1 | 0 | 0 | 0 | −2 | 0 | 10 | 0 | 1 | 0 | 2 | 0 | −2 | 0 |
| 1 | 0 | 4 | 0 | 3 | −1 | −1 | 1 | 0 | −1 | 0 | 0 | 0 | −1 | −2 | 20 | −1 | 0 | 1 | 0 | 0 | 1 | 3 |
| −1 | −1 | 4 | 0 | 0 | 0 | 1 | 0 | −2 | 3 | 0 | 0 | 0 | 1 | 2 | −20 | 0 | −1 | −1 | 0 | 2 | 1 | −1 |
| 1 | 0 | 5 | 0 | 3 | −1 | 0 | 0 | 0 | 2 | 0 | 0 | 1 | −1 | −1 | 5 | 0 | 0 | −1 | 1 | 0 | 1 | −3 |
| 0 | 0 | −9 | −1 | 0 | 0 | 0 | 0 | 1 | 0 | 0 | 0 | −1 | 0 | 3 | −15 | 0 | 0 | 0 | 1 | 2 | 0 | 0 |
| −1 | 0 | 2 | 0 | 0 | 0 | 1 | 0 | 0 | −3 | 0 | 0 | 0 | 1 | −2 | 10 | −1 | 0 | 1 | −2 | 0 | 1 | 1 |
| 2 | −1 | 0 | 1 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | −2 | −2 | −2 |
| 1 | 0 | 2 | −1 | −3 | 1 | 0 | 0 | 1 | −2 | 0 | 0 | 0 | 1 | −4 | 10 | 0 | −2 | −1 | 2 | 2 | 1 | 3 |
| −2 | 1 | 6 | 1 | 0 | 0 | −1 | 0 | 1 | −3 | 0 | −1 | 0 | 0 | 0 | −6 | 0 | 1 | 0 | 2 | 1 | 2 | −2 |
| −1 | 0 | −2 | 0 | −1 | 1 | 1 | 0 | 0 | 1 | −1 | 0 | 0 | 1 | −2 | 10 | 0 | 0 | 1 | −2 | 0 | −1 | −1 |
| 2 | 0 | 9 | 0 | 0 | 0 | −1 | 1 | 0 | −3 | 0 | 0 | −1 | 0 | 3 | −15 | 1 | 0 | 0 | 1 | 0 | −2 | 2 |
| −1 | 1 | −5 | 1 | 0 | 0 | −1 | 0 | −1 | 3 | 0 | 0 | 1 | −1 | −1 | 5 | 0 | −1 | −1 | 1 | 0 | 1 | 1 |
| −1 | 1 | 2 | −1 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | 0 | −1 | 4 | −10 | −1 | 1 | 1 | −2 | 0 | 1 | −1 |
| −1 | 1 | −4 | 0 | 3 | −1 | 1 | 0 | 0 | −1 | 0 | 1 | 0 | −1 | 2 | −4 | 0 | −1 | −1 | 0 | −1 | −1 | 3 |
| 2 | −1 | 0 | −1 | −2 | 0 | 0 | −1 | 1 | 0 | 1 | 0 | 0 | 0 | 0 | 0 | −1 | 1 | 0 | 0 | 0 | 2 | −2 |
| −2 | 0 | 3 | 0 | 3 | −1 | 1 | 0 | 0 | −1 | 0 | 0 | −1 | 2 | −3 | 5 | 1 | 0 | 0 | 1 | 0 | −2 | 0 |
| 1 | 0 | −4 | 0 | 0 | 0 | −1 | 1 | −2 | 3 | 0 | −1 | 0 | 1 | −2 | 4 | 1 | 0 | 1 | 0 | 1 | −1 | −1 |
| 0 | −1 | 3 | 1 | −1 | 1 | 0 | 0 | 1 | −2 | −1 | 0 | 1 | −2 | 3 | −5 | 0 | −1 | 0 | −1 | 0 | 0 | 2 |
| −1 | 1 | 1 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| −1 | 1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 |

*Tables of Characters of Transitive Subgroups. Alternating Groups*

The simple characters of the symmetric groups which are not self-associated are also simple characters of the alternating groups.

Every self-associated character of the symmetric group (marked with a † in the tables) is the sum of two simple characters of the alternating group. These characters of the alternating group take exactly half the values of the character of the symmetric group, save for that class for which the character of the symmetric group is $\pm 1$. This class splits into two for the alternating group, and it is for these classes alone that the two characters of the alternating group differ, the characteristics of the two classes being interchanged for the second character.

We give the corresponding characteristics of these classes for the alternating groups of degrees up to 9, since this is all that is necessary to complete the table of characters, when the characters of the symmetric groups are known.

*Degree* 3.

Character $[2\,1]$. Class $(3)$. Values $\frac{1}{2}(-1\pm i\sqrt{3})$.

*Degree* 4.

Character $[2^2]$. Class $(1\,3)$. Values $\frac{1}{2}(-1\pm i\sqrt{3})$.

*Degree* 5.

Character $[3\,1^2]$. Class $(5)$. Values $\frac{1}{2}(1\pm\sqrt{5})$.

*Degree* 6.

Character $[3\,2\,1]$. Class $(1\,5)$. Values $\frac{1}{2}(1\pm\sqrt{5})$.

*Degree* 7.

Character $[4\,1^3]$. Class $(7)$. Values $\frac{1}{2}(-1\pm i\sqrt{7})$.

*Degree* 8.

Character $[4\,2\,1^2]$. Class $(1\,7)$. Values $\frac{1}{2}(-1\pm i\sqrt{7})$.
Character $[3^2\,2]$. Class $(3\,5)$. Values $\frac{1}{2}(-1\pm i\sqrt{15})$.

*Degree* 9.

Character $[5\,1^4]$. Class $(9)$. Values $\frac{1}{2}(1\pm 3) = 2$ and $-1$.
Character $[3^3]$. Class $(1\,3\,5)$. Values $\frac{1}{2}(-1\pm i\sqrt{15})$.

If $n = \lambda_1+\lambda_2+\lambda_3+\ldots$ is a self-associated partition of $n$, then

$$n = p_1+p_2+p_3+\ldots,$$

where

$$p_1 = 2\lambda_1-1,$$
$$p_2 = 2\lambda_2-3,$$
$$p_3 = 2\lambda_3-5,$$

is a partition of $n$ into odd numbers. The characteristic of the class $(p_1 p_2 p_3 \ldots)$ of the symmetric group, corresponding to the partition

$$n = \lambda_1+\lambda_2+\ldots$$

of $n$, is given by $$\chi_p^{(\lambda)} = (-1)^{\frac{1}{2}(p_1 p_2 p_3\ldots-1)}.$$

The corresponding characteristics of the two classes of the alternating group into which this class separates are

$$\frac{(-1)^{\frac{1}{2}(p_1 p_2\ldots-1)}\pm\sqrt{(p_1 p_2 p_3\ldots)}(i)^{\frac{1}{2}(p_1 p_2\ldots-1)}}{2}.$$

Frobenius[†] proves these properties, which may also be deduced from the orthogonal properties.

### *General Cyclic Group of Order n*

Let $S$ be any operation of the group whose order is $n$, and let $\omega$ be a primitive $n$th root of unity.

There is a character $\chi$ taking the value $\omega^r$ for the operation $S^r$. The $n$ distinct powers of this character give the $n$ characters of the group.

### *Other Transitive Subgroups*

*Degree* 4.

Subgroup of order 8. Compound character $[4]+[2^2]$.

| Cycles | $1^4$ | $1^2 2$ | $4$ | $2^2$ | $2^2$ |
|---|---|---|---|---|---|
| Order | 1 | 2 | 2 | 2 | 1 |
| | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | −1 | 1 | 1 |
| | 1 | 1 | −1 | −1 | 1 |
| | 1 | −1 | 1 | −1 | 1 |
| | 2 | 0 | 0 | 0 | −2 |

---

[†] Frobenius (35).

Self-conjugate subgroup of order 4. Compound character $[4]+2[2^2]+[1^4]$.

| Cycles | $1^4$ | $2^2$ | $2^2$ | $2^2$ |
|---|---|---|---|---|
| Order | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 |
| | 1 | 1 | $-1$ | $-1$ |
| | 1 | $-1$ | 1 | $-1$ |
| | 1 | $-1$ | $-1$ | 1 |

*Degree* 5.

Subgroup of order 20. Compound character $[5]+[2^2 1]$.

| Cycles | $1^5$ | $1\,4$ | $1\,4$ | $1\,2^2$ | $5$ |
|---|---|---|---|---|---|
| Order | 1 | 5 | 5 | 5 | 4 |
| | 1 | 1 | 1 | 1 | 1 |
| | 1 | $-1$ | $-1$ | 1 | 1 |
| | 1 | $-i$ | $i$ | $-1$ | 1 |
| | 1 | $i$ | $-i$ | $-1$ | 1 |
| | 4 | 0 | 0 | 0 | $-1$ |

Subgroup of order 10 of above group.

| Cycles | $1^5$ | $1\,2^2$ | $5$ | $5$ |
|---|---|---|---|---|
| Order | 1 | 5 | 2 | 2 |
| | 1 | 1 | 1 | 1 |
| | 1 | $-1$ | 1 | 1 |
| | 2 | 0 | $-\frac{1}{2}(1+\sqrt{5})$ | $-\frac{1}{2}(1-\sqrt{5})$ |
| | 2 | 0 | $-\frac{1}{2}(1-\sqrt{5})$ | $-\frac{1}{2}(1+\sqrt{5})$ |

*Degree* 6.

Subgroup of order 120. Compound character $[6]+[2^3]$.

| Cycles | $1^6$ | $1^2\,4$ | $1^2\,2^2$ | $1\,5$ | $6$ | $2^3$ | $3^2$ |
|---|---|---|---|---|---|---|---|
| Order | 1 | 30 | 15 | 24 | 20 | 10 | 20 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 4 | 0 | 0 | $-1$ | $-1$ | 2 | 1 |
| | 5 | $-1$ | 1 | 0 | 1 | 1 | $-1$ |
| | 6 | 0 | $-2$ | 1 | 0 | 0 | 0 |
| | 5 | 1 | 1 | 0 | $-1$ | $-1$ | $-1$ |
| | 4 | 0 | 0 | $-1$ | 1 | $-2$ | 1 |
| | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 |

This group is clearly simply isomorphic with the symmetric group of order 5!. The characters of subgroups of this group, then, may be obtained from the characters of groups of degree 5.

Subgroup of order 72. Compound character [6]+[4 2].

| Cycles | $1^6$ | $1^4\,2$ | $1^3\,3$ | $1^2\,2^2$ | 1 2 3 | 6 | 2 4 | $2^3$ | $3^2$ |
|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 6 | 4 | 9 | 12 | 12 | 18 | 6 | 4 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | 1 | 1 | −1 | −1 | 1 | −1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | 1 |
| | 1 | −1 | 1 | 1 | −1 | 1 | −1 | 1 | 1 |
| | 4 | 2 | 1 | 0 | −1 | 0 | 0 | 0 | −2 |
| | 4 | −2 | 1 | 0 | 1 | 0 | 0 | 0 | −2 |
| | 4 | 0 | −2 | 0 | 0 | −1 | 0 | 2 | 1 |
| | 4 | 0 | −2 | 0 | 0 | 1 | 0 | −2 | 1 |
| | 2 | 0 | 2 | −2 | 0 | 0 | 0 | 0 | 2 |

Subgroup of order 48. Compound character [6]+[4 2]+$[2^3]$.

| Cycles | $1^6$ | $1^4\,2$ | $1^2\,4$ | $1^2\,2^2$ | $1^2\,2^2$ | 6 | 2 4 | $2^3$ | $2^3$ | $3^2$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 3 | 6 | 3 | 6 | 8 | 6 | 6 | 1 | 8 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | −1 | 1 | 1 | −1 | 1 | −1 | −1 | 1 |
| | 1 | 1 | −1 | 1 | −1 | 1 | −1 | −1 | 1 | 1 |
| | 1 | −1 | 1 | 1 | −1 | −1 | −1 | 1 | −1 | 1 |
| | 2 | 2 | 0 | 2 | 0 | −1 | 0 | 0 | 2 | −1 |
| | 2 | −2 | 0 | 2 | 0 | 1 | 0 | 0 | −2 | −1 |
| | 3 | 1 | 1 | −1 | 1 | 0 | −1 | −1 | −3 | 0 |
| | 3 | −1 | −1 | −1 | 1 | 0 | −1 | 1 | 3 | 0 |
| | 3 | 1 | −1 | −1 | −1 | 0 | 1 | 1 | −3 | 0 |
| | 3 | −1 | 1 | −1 | −1 | 0 | 1 | −1 | 3 | 0 |

*Degree* 7.

Subgroup of order 168. Compound character [7]+[4 3]+$[2^3 1]$+$[1^7]$.

| Cycles | $1^7$ | $1^3\,2^2$ | 1 2 4 | $1\,3^2$ | 7 | 7 |
|---|---|---|---|---|---|---|
| Order | 1 | 21 | 42 | 56 | 24 | 24 |
| | 1 | 1 | 1 | 1 | 1 | 1 |
| | 6 | 2 | 0 | 0 | −1 | −1 |
| | 7 | −1 | −1 | 1 | 0 | 0 |
| | 8 | 0 | 0 | −1 | 1 | 1 |
| | 3 | −1 | 1 | 0 | $\frac{1}{2}(-1+i\sqrt{7})$ | $\frac{1}{2}(-1-i\sqrt{7})$ |
| | 3 | −1 | 1 | 0 | $\frac{1}{2}(-1-i\sqrt{7})$ | $\frac{1}{2}(-1+i\sqrt{7})$ |

Subgroup of order 42. Compound character
$[7]+[4\,3]+[4\,2\,1]+[4\,1^3]+[3\,2^2]+[2^3\,1]+[3\,1^4]$.

| Cycles | $1^7$ | $1\,6$ | $1\,6$ | $1\,2^3$ | $1\,3^2$ | $1\,3^2$ | $7$ |
|---|---|---|---|---|---|---|---|
| Order | 1 | 7 | 7 | 7 | 7 | 7 | 6 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | $\omega$ | $\omega^2$ | 1 | $\omega$ | $\omega^2$ | 1 |
| | 1 | $\omega^2$ | $\omega$ | 1 | $\omega^2$ | $\omega$ | 1 |
| | 1 | $-\omega$ | $-\omega^2$ | $-1$ | $\omega$ | $\omega^2$ | 1 |
| | 1 | $-\omega^2$ | $-\omega$ | $-1$ | $\omega^2$ | $\omega$ | 1 |
| | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 |
| | 6 | 0 | 0 | 0 | 0 | 0 | $-1$ |

($\omega$ is a complex cube root of unity.)

Subgroup of order 21 of above groups.

| Cycles | $1^7$ | $1\,3^2$ | $1\,3^2$ | $7$ | $7$ |
|---|---|---|---|---|---|
| Order | 1 | 7 | 7 | 3 | 3 |
| | 1 | 1 | 1 | 1 | 1 |
| | 1 | $\omega$ | $\omega^2$ | 1 | 1 |
| | 1 | $\omega^2$ | $\omega$ | 1 | 1 |
| | 3 | 0 | 0 | $\frac{1}{2}(-1+i\sqrt{7})$ | $\frac{1}{2}(-1-i\sqrt{7})$ |
| | 3 | 0 | 0 | $\frac{1}{2}(-1-i\sqrt{7})$ | $\frac{1}{2}(-1+i\sqrt{7})$ |

## *Degree* 8.

Subgroup of order 1344. Compound character $[8]+[4^2]+[2^4]+[1^8]$.

| Cycles | $1^8$ | $1^4\,2^2$ | $2^4$ | $2^4$ | $1^2\,3^2$ | $1^2\,2\,4$ | $2\,6$ | $1\,7$ | $1\,7$ | $4^2$ | $4^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 42 | 42 | 7 | 224 | 168 | 224 | 192 | 192 | 168 | 84 |
| (*a*) | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| (*b*) | 6 | 2 | 2 | 6 | 0 | 0 | 0 | $-1$ | $-1$ | 0 | 2 |
| (*c*) | 7 | 3 | $-1$ | $-1$ | 1 | 1 | $-1$ | 0 | 0 | $-1$ | $-1$ |
| (*d*) | 14 | 2 | 2 | $-2$ | $-1$ | 0 | 1 | 0 | 0 | 0 | $-2$ |
| (*e*) | 21 | 1 | $-3$ | $-3$ | 0 | $-1$ | 0 | 0 | 0 | 1 | 1 |
| (*f*) | 7 | $-1$ | 3 | $-1$ | 1 | $-1$ | $-1$ | 0 | 0 | 1 | $-1$ |
| (*g*) | 21 | $-3$ | 1 | $-3$ | 0 | 1 | 0 | 0 | 0 | $-1$ | 1 |
| (*h*) | 7 | $-1$ | $-1$ | 7 | 1 | $-1$ | 1 | 0 | 0 | $-1$ | $-1$ |
| (*j*) | 8 | 0 | 0 | 8 | $-1$ | 0 | $-1$ | 1 | 1 | 0 | 0 |
| (*k*) | 3 | $-1$ | $-1$ | 3 | 0 | 1 | 0 | $\frac{1}{2}(-1-i\sqrt{7})$ | $\frac{1}{2}(-1+i\sqrt{7})$ | 1 | $-1$ |
| (*l*) | 3 | $-1$ | $-1$ | 3 | 0 | 1 | 0 | $\frac{1}{2}(-1+i\sqrt{7})$ | $\frac{1}{2}(-1-i\sqrt{7})$ | 1 | $-1$ |

## Subgroup of order 1152. Compound character $[8]+[6\,2]+[4^2]$.

| Cycles | $1^8$ | $1^6 2$ | $1^5 3$ | $1^4 4$ | $1^2 2^3$ | $1^4 2^2$ | $1^3 2 3$ | $1^2 2 4$ | $1^4 2^2$ | $1^2 3^2$ | $1 2^2 3$ | $1 3 4$ | $8$ | $4^2$ | $4^2$ | $2^2 4$ | $2^2 4$ | $2 6$ | $2^4$ | $2^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 12 | 16 | 12 | 36 | 6 | 96 | 72 | 36 | 64 | 48 | 96 | 144 | 72 | 36 | 144 | 36 | 192 | 9 | 24 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | −1 | −1 | 1 | −1 | 1 | −1 | 1 | −1 |
| | 1 | −1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | 1 | 1 | −1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 | −1 |
| | 6 | 4 | 3 | 2 | 0 | 2 | 1 | 0 | 2 | 0 | −1 | −1 | 0 | 0 | −2 | 0 | −2 | 0 | −2 | 0 |
| | 6 | −4 | 3 | −2 | 0 | 2 | −1 | 0 | 2 | 0 | −1 | 1 | 0 | 0 | −2 | 0 | 2 | 0 | −2 | 0 |
| | 2 | 0 | 2 | 0 | 0 | 2 | 0 | −2 | −2 | 2 | 2 | 0 | 0 | 0 | −2 | 0 | 0 | 0 | 2 | 0 |
| | 18 | 0 | 0 | 0 | 0 | −6 | 0 | 2 | −2 | 0 | 0 | 0 | 0 | 0 | −2 | 0 | 0 | 0 | 2 | 0 |
| | 12 | 2 | −3 | −2 | 2 | 4 | −1 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | 0 | 0 | −2 | 0 | −4 | 0 |
| | 12 | −2 | −3 | 2 | −2 | 4 | 1 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | 0 | 0 | 2 | 0 | −4 | 0 |
| | 9 | 3 | 0 | −3 | −1 | −3 | 0 | −1 | 1 | 0 | 0 | 0 | 1 | 1 | 1 | −1 | 1 | 0 | 1 | −3 |
| | 9 | −3 | 0 | 3 | 1 | −3 | 0 | −1 | 1 | 0 | 0 | 0 | −1 | 1 | 1 | 1 | −1 | 0 | 1 | −3 |
| | 9 | 3 | 0 | −3 | −1 | −3 | 0 | −1 | 1 | 0 | 0 | 0 | −1 | −1 | 1 | 1 | 1 | 0 | 1 | 3 |
| | 9 | −3 | 0 | 3 | 1 | −3 | 0 | −1 | 1 | 0 | 0 | 0 | 1 | −1 | 1 | −1 | −1 | 0 | 1 | 3 |
| | 4 | 2 | 1 | 2 | 2 | 4 | −1 | 0 | 0 | −2 | 1 | −1 | 0 | 0 | 0 | 0 | 2 | 0 | 4 | 0 |
| | 4 | −2 | 1 | −2 | −2 | 4 | 1 | 0 | 0 | −2 | 1 | 1 | 0 | 0 | 0 | 0 | −2 | 0 | 4 | 0 |
| | 4 | 0 | −2 | 0 | 0 | 4 | 0 | 0 | 0 | 1 | −2 | 0 | 0 | 2 | 0 | 0 | 0 | −1 | 4 | 2 |
| | 4 | 0 | −2 | 0 | 0 | 4 | 0 | 0 | 0 | 1 | −2 | 0 | 0 | −2 | 0 | 0 | 0 | 1 | 4 | −2 |
| | 6 | 2 | 3 | 4 | −2 | 2 | −1 | 0 | −2 | 0 | −1 | 1 | 0 | 0 | 2 | 0 | 0 | 0 | −2 | 0 |
| | 6 | −2 | 3 | −4 | 2 | 2 | 1 | 0 | −2 | 0 | −1 | −1 | 0 | 0 | 2 | 0 | 0 | 0 | −2 | 0 |

## Subgroup† of order 192 of above group of order 1344. Compound character $[8]+[6\,2]+2[4^2]+[4\,2^2]+[3^2 1^2]+2[2^4]+[2^2 1^4]+[1^8]$.

| Cycles | $1^8$ | $1^4 2^2$ | $1^4 2^2$ | $2^4$ | $2^4$ | $2^4$ | $2^4$ | $1^2 3^2$ | $1^2 2 4$ | $2 6$ | $4^2$ | $4^2$ | $4^2$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 12 | 6 | 12 | 6 | 1 | 6 | 32 | 24 | 32 | 24 | 24 | 12 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 3 | 1 | −1 | 1 | −1 | 3 | 3 | 0 | −1 | 0 | −1 | 1 | −1 |
| | 2 | 0 | 2 | 0 | 2 | 2 | 2 | −1 | 0 | −1 | 0 | 0 | 2 |
| | 3 | −1 | −1 | −1 | −1 | 3 | 3 | 0 | 1 | 0 | 1 | −1 | −1 |
| | 1 | −1 | 1 | −1 | 1 | 1 | 1 | 1 | −1 | 1 | −1 | −1 | 1 |
| | 3 | −1 | 3 | −1 | −1 | 3 | −1 | 0 | −1 | 0 | 1 | 1 | −1 |
| | 3 | 1 | 3 | 1 | −1 | 3 | −1 | 0 | 1 | 0 | −1 | −1 | −1 |
| | 3 | 1 | −1 | 1 | 3 | 3 | −1 | 0 | −1 | 0 | 1 | −1 | −1 |
| | 3 | −1 | −1 | −1 | 3 | 3 | −1 | 0 | 1 | 0 | −1 | 1 | −1 |
| | 6 | 0 | −2 | 0 | −2 | 6 | −2 | 0 | 0 | 0 | 0 | 0 | 2 |
| | 4 | 2 | 0 | −2 | 0 | −4 | 0 | 1 | 0 | −1 | 0 | 0 | 0 |
| | 4 | −2 | 0 | 2 | 0 | −4 | 0 | 1 | 0 | −1 | 0 | 0 | 0 |
| | 8 | 0 | 0 | 0 | 0 | 8 | 0 | −1 | 0 | 1 | 0 | 0 | 0 |

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† This group is the positive group of the group of order 384 given below. The characters of the latter group may be found by the method given in the text, but an easier method is to find the self-associated characters of the symmetric group corresponding to the simple characters of the above group. These may be divided into two associated compound characters which correspond to simple characters of the group of order 384. The latter group is a maximal subgroup of the symmetric group.

Subgroup of order 384. Compound character $[8]+[6\,2]+[4^2]+[4\,2^2]+[2^4]$.

| Cycles | $1^8$ | $1^6\,2$ | $1^4\,4$ | $1^4\,2^2$ | $1^4\,2^2$ | $1\,6$ | $1^2\,2\,4$ | $1^2\,2^3$ | $1^2\,2^3$ | $1^2\,3^2$ | $8$ | $4^2$ | $4^2$ | $2^2\,4$ | $2^2\,4$ | $2\,6$ | $2\,3^2$ | $2^4$ | $2^4$ | $2^4$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 4 | 12 | 12 | 6 | 32 | 24 | 24 | 4 | 32 | 48 | 48 | 12 | 24 | 12 | 32 | 32 | 12 | 12 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | −1 | 1 | 1 | −1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | 1 |
| | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | 1 | 1 | −1 | 1 | 1 |
| | 1 | −1 | 1 | −1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | −1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 |
| | 2 | 2 | 0 | 0 | 2 | −1 | 0 | 0 | 2 | −1 | 0 | 0 | 2 | 2 | 0 | −1 | −1 | 0 | 2 | 2 |
| | 2 | −2 | 0 | 0 | 2 | 1 | 0 | 0 | −2 | −1 | 0 | 0 | 2 | −2 | 0 | −1 | 1 | 0 | 2 | 2 |
| | 6 | 0 | 0 | 2 | −2 | 0 | −2 | 0 | 0 | 0 | 0 | 0 | −2 | 0 | 0 | 0 | 0 | 2 | 2 | 6 |
| | 6 | 0 | 0 | −2 | −2 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | −2 | 0 | 0 | 0 | 0 | −2 | 2 | 6 |
| | 3 | 3 | 1 | 1 | 3 | 0 | 1 | 1 | 3 | 0 | −1 | −1 | −1 | −1 | 1 | 0 | 0 | 1 | −1 | 3 |
| | 3 | −3 | −1 | 1 | 3 | 0 | 1 | −1 | −3 | 0 | 1 | −1 | −1 | 1 | −1 | 0 | 0 | 1 | −1 | 3 |
| | 3 | 3 | −1 | −1 | 3 | 0 | −1 | −1 | 3 | 0 | 1 | 1 | −1 | −1 | −1 | 0 | 0 | −1 | −1 | 3 |
| | 3 | −3 | 1 | −1 | 3 | 0 | −1 | 1 | −3 | 0 | −1 | 1 | −1 | 1 | 1 | 0 | 0 | −1 | −1 | 3 |
| | 4 | 2 | 2 | 2 | 0 | 1 | 0 | 0 | −2 | 1 | 0 | 0 | 0 | 0 | −2 | −1 | −1 | −2 | 0 | −4 |
| | 4 | −2 | −2 | 2 | 0 | −1 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | 2 | −1 | 1 | −2 | 0 | −4 |
| | 4 | 2 | −2 | −2 | 0 | 1 | 0 | 0 | −2 | 1 | 0 | 0 | 0 | 0 | 2 | −1 | −1 | 2 | 0 | −4 |
| | 4 | −2 | 2 | −2 | 0 | −1 | 0 | 0 | 2 | 1 | 0 | 0 | 0 | 0 | −2 | −1 | 1 | 2 | 0 | −4 |
| | 6 | 0 | 2 | 0 | −2 | 0 | 0 | −2 | 0 | 0 | 0 | 0 | 2 | 0 | 2 | 0 | 0 | 0 | −2 | 6 |
| | 6 | 0 | −2 | 0 | −2 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | 2 | 0 | −2 | 0 | 0 | 0 | −2 | 6 |
| | 8 | 4 | 0 | 0 | 0 | −1 | 0 | 0 | −4 | −1 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | 0 | 0 | −8 |
| | 8 | −4 | 0 | 0 | 0 | 1 | 0 | 0 | 4 | −1 | 0 | 0 | 0 | 0 | 0 | 1 | −1 | 0 | 0 | −8 |

Subgroup of order 168 of above group of order 1344. Compound character $[8]+[5\,1^3]+[4^2]+[4\,3\,1]+[4\,1^4]+[3\,2^2\,1]+[2^4]+[1^8]$.

| Cycles | $1^8$ | $2^4$ | $1^2\,3^2$ | $1^2\,3^2$ | $2\,6$ | $2\,6$ | $1\,7$ | $1\,7$ |
|---|---|---|---|---|---|---|---|---|
| Order | 1 | 7 | 28 | 28 | 28 | 28 | 24 | 24 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 3 | 3 | 0 | 0 | 0 | 0 | $\frac{1}{2}(-1+i\sqrt{7})$ | $\frac{1}{2}(-1-i\sqrt{7})$ |
| | 3 | 3 | 0 | 0 | 0 | 0 | $\frac{1}{2}(-1-i\sqrt{7})$ | $\frac{1}{2}(-1+i\sqrt{7})$ |
| | 7 | −1 | 1 | 1 | −1 | −1 | 0 | 0 |
| | 7 | −1 | $\omega$ | $\omega^2$ | $-\omega$ | $-\omega^2$ | 0 | 0 |
| | 7 | −1 | $\omega^2$ | $\omega$ | $-\omega^2$ | $-\omega$ | 0 | 0 |
| | 1 | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | 1 | 1 |
| | 1 | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | 1 | 1 |

($\omega$ is a complex cube root of unity.)

Subgroup† of order 168 of above group of order 1344. Compound character $[8]+[5\,1^3]+2[4^2]+[4\,2^2]+[4\,1^4]+[3^2\,1^2]+2[2^4]+[1^8]$.

| Cycles | $1^8$ | $2^4$ | $1^2\,3^2$ | $4^2$ | $1\,7$ | $1\,7$ |
|---|---|---|---|---|---|---|
| Order | 1 | 21 | 56 | 42 | 24 | 24 |
| | 1 | 1 | 1 | 1 | 1 | 1 |
| | 6 | 2 | 0 | 0 | −1 | −1 |
| | 7 | −1 | 1 | −1 | 0 | 0 |
| | 8 | 0 | −1 | 0 | 1 | 1 |
| | 3 | −1 | 0 | 1 | $\frac{1}{2}(-1+i\sqrt{7})$ | $\frac{1}{2}(-1-i\sqrt{7})$ |
| | 3 | −1 | 0 | 1 | $\frac{1}{2}(-1-i\sqrt{7})$ | $\frac{1}{2}(-1+i\sqrt{7})$ |

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† This group is the positive group of the group of order 336 given below, of which the characters may be found by the same method as for the group of order 384.

Subgroup of order 336. Compound character

$[8]+[4^2]+[4\,2^2]+[4\,1^4]+[2^4]$.

| Cycles | $1^8$ | $2^4$ | $1^2\,3^2$ | $4^2$ | $1\,7$ | $1^2\,6$ | $1^2\,2^3$ | $8$ | $8$ |
|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 21 | 56 | 42 | 48 | 56 | 28 | 42 | 42 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ |
| | 6 | 2 | 0 | 0 | $-1$ | 0 | 0 | $\sqrt{2}$ | $-\sqrt{2}$ |
| | 6 | 2 | 0 | 0 | $-1$ | 0 | 0 | $-\sqrt{2}$ | $\sqrt{2}$ |
| | 7 | $-1$ | 1 | $-1$ | 0 | 1 | 1 | $-1$ | $-1$ |
| | 7 | $-1$ | 1 | $-1$ | 0 | $-1$ | $-1$ | 1 | 1 |
| | 8 | 0 | $-1$ | 0 | 1 | $-1$ | 2 | 0 | 0 |
| | 8 | 0 | $-1$ | 0 | 1 | 1 | $-2$ | 0 | 0 |
| | 6 | $-2$ | 0 | 2 | $-1$ | 0 | 0 | 0 | 0 |

*Degree* 9.

Subgroup of order 1512. Compound character

$[9]+[4^2\,1]+[5\,1^4]+[3\,2^3]+[1^9]$.

| Cycles | $1^9$ | $1^3\,3^2$ | $1^3\,3^2$ | $1^2\,7$ | $1\,2\,6$ | $1\,2\,6$ | $1\,2^4$ | $3^3$ | $9$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 84 | 84 | 216 | 252 | 252 | 63 | 56 | 168 | 168 | 168 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | $\omega$ | $\omega^2$ | 1 | $\omega$ | $\omega^2$ | 1 | 1 | 1 | $\omega$ | $\omega^2$ |
| | 1 | $\omega^2$ | $\omega$ | 1 | $\omega^2$ | $\omega$ | 1 | 1 | 1 | $\omega^2$ | $\omega$ |
| | 7 | 1 | 1 | 0 | $-1$ | $-1$ | $-1$ | $-2$ | 1 | 1 | 1 |
| | 7 | $\omega$ | $\omega^2$ | 0 | $-\omega$ | $-\omega^2$ | $-1$ | $-2$ | 1 | $\omega$ | $\omega^2$ |
| | 7 | $\omega^2$ | $\omega$ | 0 | $-\omega^2$ | $-\omega$ | $-1$ | $-2$ | 1 | $\omega^2$ | $\omega$ |
| | 8 | 2 | 2 | 1 | 0 | 0 | 0 | $-1$ | $-1$ | $-1$ | $-1$ |
| | 8 | $2\omega$ | $2\omega^2$ | 1 | 0 | 0 | 0 | $-1$ | $-1$ | $-\omega$ | $-\omega^2$ |
| | 8 | $2\omega^2$ | $2\omega$ | 1 | 0 | 0 | 0 | $-1$ | $-1$ | $-\omega^2$ | $-\omega$ |
| | 21 | 0 | 0 | 0 | 0 | 0 | $-3$ | 3 | 0 | 0 | 0 |
| | 27 | 0 | 0 | $-1$ | 0 | 0 | 3 | 0 | 0 | 0 | 0 |

Subgroup of order 504 of above group.

| Cycles | $1^9$ | $1^2\,7$ | $1^2\,7$ | $1^2\,7$ | $1\,2^4$ | $3^3$ | $9$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 72 | 72 | 72 | 63 | 56 | 56 | 56 | 56 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 7 | 0 | 0 | 0 | $-1$ | $-2$ | 1 | 1 | 1 |
| | 7 | 0 | 0 | 0 | $-1$ | 1 | $-2$ | 1 | 1 |
| | 7 | 0 | 0 | 0 | $-1$ | 1 | 1 | $-2$ | 1 |
| | 7 | 0 | 0 | 0 | $-1$ | 1 | 1 | 1 | $-2$ |
| | 9 | 0 | $-2$ | 1 | 1 | 0 | 0 | 0 | 0 |
| | 9 | $-2$ | 1 | 0 | 1 | 0 | 0 | 0 | 0 |
| | 9 | 1 | 0 | $-2$ | 1 | 0 | 0 | 0 | 0 |
| | 8 | 1 | 1 | 1 | 0 | $-1$ | $-1$ | $-1$ | $-1$ |

Subgroup of order 1296. Compound character
$[9]+[7\,2]+[6\,3]+[5\,2^2]+[4^2\,1]$.

| Cycles | $1^9$ | $1^7 2$ | $1^6 3$ | $1^5 2^2$ | $1^4 2 3$ | $1^3 6$ | $1^3 2 4$ | $1^3 2^3$ | $1^3 2^3$ | $1^3 3^2$ | $1^2 2^2 3$ | $1 2^2 4$ | $1 2 6$ | $1 2 3^2$ | $1 2^4$ | $3^3$ | $3^3$ | $3 6$ | $3 6$ | $2 3 4$ | $2^3 3$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 9 | 6 | 27 | 36 | 36 | 54 | 18 | 27 | 12 | 54 | 162 | 108 | 36 | 54 | 8 | 72 | 72 | 216 | 108 | 36 | 144 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | 1 | 1 | −1 | −1 | 1 | −1 | −1 | 1 | 1 | −1 | 1 | −1 | 1 | 1 | 1 | −1 | −1 | 1 | −1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 | −1 | −1 | 1 | −1 | 1 | 1 | −1 | 1 | −1 | −1 | 1 |
| | 1 | −1 | 1 | 1 | −1 | 1 | −1 | 1 | −1 | 1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 | −1 | −1 | 1 | 1 |
| | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 2 | 2 | 2 | 0 | 0 | 2 | 0 | 2 | −1 | 0 | −1 | 0 | 0 | −1 |
| | 2 | −2 | 2 | 2 | −2 | 0 | 0 | 0 | −2 | 2 | 2 | 0 | 0 | −2 | 0 | 2 | −1 | 0 | 1 | 0 | 0 | −1 |
| | 6 | 4 | 3 | 2 | 1 | 2 | 2 | 2 | 0 | 0 | −1 | 0 | 0 | −2 | 0 | −3 | 0 | −1 | 0 | −1 | −1 | 0 |
| | 6 | −4 | 3 | 2 | −1 | −2 | 2 | −2 | 0 | 0 | −1 | 0 | 0 | 2 | 0 | −3 | 0 | 1 | 0 | −1 | 1 | 0 |
| | 6 | 4 | 3 | 2 | 1 | −2 | −2 | −2 | 0 | 0 | −1 | 0 | 0 | −2 | 0 | −3 | 0 | 1 | 0 | 1 | 1 | 0 |
| | 6 | −4 | 3 | 2 | −1 | 2 | −2 | 2 | 0 | 0 | −1 | 0 | 0 | 2 | 0 | −3 | 0 | −1 | 0 | 1 | −1 | 0 |
| | 12 | 4 | 0 | 0 | −2 | 1 | 0 | −2 | 0 | −3 | 0 | 0 | 1 | 1 | −2 | 3 | 0 | 1 | 0 | 0 | −2 | 0 |
| | 12 | −4 | 0 | 0 | 2 | −1 | 0 | 2 | 0 | −3 | 0 | 0 | 1 | −1 | −2 | 3 | 0 | −1 | 0 | 0 | 2 | 0 |
| | 12 | 4 | 0 | 0 | −2 | −1 | 0 | 2 | 0 | −3 | 0 | 0 | −1 | 1 | 2 | 3 | 0 | −1 | 0 | 0 | 2 | 0 |
| | 12 | −4 | 0 | 0 | 2 | 1 | 0 | −2 | 0 | −3 | 0 | 0 | −1 | −1 | 2 | 3 | 0 | 1 | 0 | 0 | −2 | 0 |
| | 3 | −1 | 3 | −1 | −1 | −1 | 1 | −1 | 3 | 3 | −1 | 1 | −1 | −1 | −1 | 3 | 0 | −1 | 0 | 1 | −1 | 0 |
| | 3 | 1 | 3 | −1 | 1 | 1 | 1 | 1 | −3 | 3 | −1 | −1 | −1 | 1 | −1 | 3 | 0 | 1 | 0 | 1 | 1 | 0 |
| | 3 | −1 | 3 | −1 | −1 | 1 | −1 | 1 | 3 | 3 | −1 | −1 | 1 | −1 | 1 | 3 | 0 | 1 | 0 | −1 | 1 | 0 |
| | 3 | 1 | 3 | −1 | 1 | −1 | −1 | −1 | −3 | 3 | −1 | 1 | 1 | 1 | 1 | 3 | 0 | −1 | 0 | −1 | −1 | 0 |
| | 8 | 0 | −4 | 0 | 0 | 2 | 0 | −4 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | −1 | 2 | −1 | 0 | 0 | 2 | −1 |
| | 8 | 0 | −4 | 0 | 0 | −2 | 0 | 4 | 0 | 2 | 0 | 0 | 0 | 0 | 0 | −1 | 2 | 1 | 0 | 0 | −2 | −1 |
| | 12 | 0 | 6 | −4 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | 0 | 0 | −6 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 16 | 0 | −8 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 0 | 0 | 0 | 0 | 0 | −2 | −2 | 0 | 0 | 0 | 0 | 1 |

Subgroup of order 648 of above group of order 1296.
Compound character $A+B$.

| Cycles | $1^9$ | $1^6 3$ | $1^5 2^2$ | $1^3 2 4$ | $1^3 3^2$ | $1^2 2^2 3$ | $1 2 6$ | $1 2^4$ | $3^3$ | $3^3$ | $2 3 4$ | $2 3 4$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 6 | 27 | 54 | 12 | 54 | 108 | 54 | 8 | 72 | 54 | 54 | 72 | 72 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | −1 | 1 | 1 |
| | 2 | 2 | 2 | 0 | 2 | 2 | 0 | 0 | 2 | −1 | 0 | 0 | −1 | −1 |
| | 6 | 3 | 2 | 2 | 0 | −1 | 0 | 0 | −3 | 0 | −1 | −1 | 0 | 0 |
| | 6 | 3 | 2 | −2 | 0 | −1 | 0 | 0 | −3 | 0 | 1 | 1 | 0 | 0 |
| | 12 | 0 | 0 | 0 | −3 | 0 | 1 | 2 | 3 | 0 | 0 | 0 | 0 | 0 |
| | 12 | 0 | 0 | 0 | −3 | 0 | −1 | −2 | 3 | 0 | 0 | 0 | 0 | 0 |
| | 3 | 3 | −1 | 1 | 3 | −1 | −1 | −1 | 3 | 0 | 1 | 1 | 0 | 0 |
| | 3 | 3 | −1 | −1 | 3 | −1 | 1 | 1 | 3 | 0 | −1 | −1 | 0 | 0 |
| | 8 | −4 | 0 | 0 | 2 | 0 | 0 | 0 | −1 | 2 | 0 | 0 | −1 | −1 |
| | 6 | 3 | −2 | 0 | 0 | 1 | 0 | 0 | −3 | 0 | $\omega^2-\omega$ | $\omega-\omega^2$ | 0 | 0 |
| | 6 | 3 | −2 | 0 | 0 | 1 | 0 | 0 | −3 | 0 | $\omega-\omega^2$ | $\omega^2-\omega$ | 0 | 0 |
| | 8 | −4 | 0 | 0 | 2 | 0 | 0 | 0 | −1 | −1 | −1 | −1 | 2 | −1 |
| | 8 | −4 | 0 | 0 | 2 | 0 | 0 | 0 | −1 | −1 | −1 | −1 | −1 | 2 |

Subgroup of order 648 of above group of order 1296.
Compound character $A+C$.

| Cycles | $1^9$ | $1^7 2$ | $1^6 3$ | $1^5 2^2$ | $1^4 2 3$ | $1^4 2 3$ | $1^3 2^3$ | $1^3 3^2$ | $1^2 2^2 3$ | $1 2 3^2$ | $3^3$ | $3^3$ | $3^3$ | $3 6$ | $3 6$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 9 | 6 | 27 | 18 | 18 | 27 | 12 | 54 | 36 | 8 | 36 | 36 | 108 | 108 | 72 | 72 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | −1 | 1 | 1 | 1 | −1 | −1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ |
| | 1 | −1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | −1 | 1 | $\omega$ | $\omega^2$ | $-\omega$ | $-\omega^2$ | $\omega$ | $\omega^2$ |
| | 1 | −1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | −1 | 1 | $\omega^2$ | $\omega$ | $-\omega^2$ | $-\omega$ | $\omega^2$ | $\omega$ |
| | 6 | 4 | 3 | 2 | 1 | 1 | 0 | 0 | −1 | −2 | −3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 6 | −4 | 3 | 2 | −1 | −1 | 0 | 0 | −1 | 2 | −3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 12 | 4 | 0 | 0 | −2 | −2 | 0 | −3 | 0 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 12 | −4 | 0 | 0 | 2 | 2 | 0 | −3 | 0 | −1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | 1 | 3 | −1 | 1 | 1 | −3 | 3 | −1 | 1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | −1 | 3 | −1 | −1 | −1 | 3 | 3 | −1 | −1 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 6 | 0 | 3 | −2 | −3 | 3 | 0 | 0 | 1 | 0 | −3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 6 | 0 | 3 | −2 | 3 | −3 | 0 | 0 | 1 | 0 | −3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 8 | 0 | −4 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | −1 | 2 | 2 | 0 | 0 | −1 | −1 |
| | 8 | 0 | −4 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | −1 | $2\omega$ | $2\omega^2$ | 0 | 0 | $-\omega$ | $-\omega^2$ |
| | 8 | 0 | −4 | 0 | 0 | 0 | 0 | 2 | 0 | 0 | −1 | $2\omega^2$ | $2\omega$ | 0 | 0 | $-\omega^2$ | $-\omega$ |

Subgroup of order 648 of above group of order 1296.
Compound character $A+D$.

| s | $1^9$ | $1^6 3$ | $1^5 2^2$ | $1^3 6$ | $1^3 2^3$ | $1^3 3^2$ | $1^2 2^2 3$ | $1 2^2 4$ | $3^3$ | $3^3$ | $3^3$ | $3 6$ | $3 6$ | $2^3 3$ | $2^3 3$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| r | 1 | 6 | 27 | 36 | 18 | 12 | 54 | 162 | 4 | 4 | 72 | 36 | 36 | 18 | 18 | 72 | 72 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | −1 | −1 | 1 | 1 | −1 | 1 | 1 | 1 | −1 | −1 | −1 | −1 | 1 | 1 |
| | 2 | 2 | 2 | 0 | 0 | 2 | 2 | 0 | 2 | 2 | −1 | 0 | 0 | 0 | 0 | −1 | −1 |
| | 6 | 3 | 2 | 2 | 2 | 0 | −1 | 0 | −3 | −3 | 0 | −1 | −1 | −1 | −1 | 0 | 0 |
| | 6 | 3 | 2 | −2 | −2 | 0 | −1 | 0 | −3 | −3 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| | 12 | 0 | 0 | 1 | −2 | −3 | 0 | 0 | 3 | 3 | 0 | 1 | 1 | −2 | −2 | 0 | 0 |
| | 12 | 0 | 0 | −1 | 2 | −3 | 0 | 0 | 3 | 3 | 0 | −1 | −1 | 2 | 2 | 0 | 0 |
| | 3 | 3 | −1 | 1 | 1 | 3 | −1 | −1 | 3 | 3 | 0 | 1 | 1 | 1 | 1 | 0 | 0 |
| | 3 | 3 | −1 | −1 | −1 | 3 | −1 | 1 | 3 | 3 | 0 | −1 | −1 | −1 | −1 | 0 | 0 |
| | 4 | −2 | 0 | 1 | −2 | 1 | 0 | 0 | $1+3\omega$ | $1+3\omega^2$ | 1 | $\omega$ | $\omega^2$ | $-2\omega$ | $-2\omega^2$ | $\omega$ | $\omega^2$ |
| | 4 | −2 | 0 | 1 | −2 | 1 | 0 | 0 | $1+3\omega^2$ | $1+3\omega$ | 1 | $\omega^2$ | $\omega$ | $-2\omega^2$ | $-2\omega$ | $\omega^2$ | $\omega$ |
| | 4 | −2 | 0 | −1 | 2 | 1 | 0 | 0 | $1+3\omega$ | $1+3\omega^2$ | 1 | $-\omega$ | $-\omega^2$ | $2\omega$ | $2\omega^2$ | $\omega$ | $\omega^2$ |
| | 4 | −2 | 0 | −1 | 2 | 1 | 0 | 0 | $1+3\omega^2$ | $1+3\omega$ | 1 | $-\omega^2$ | $-\omega$ | $2\omega^2$ | $2\omega$ | $\omega^2$ | $\omega$ |
| | 6 | 3 | −2 | 0 | 0 | 0 | 1 | 0 | −3 | −3 | 0 | $-\sqrt{3}$ | $\sqrt{3}$ | $\sqrt{3}$ | $-\sqrt{3}$ | 0 | 0 |
| | 6 | 3 | −2 | 0 | 0 | 0 | 1 | 0 | −3 | −3 | 0 | $\sqrt{3}$ | $-\sqrt{3}$ | $-\sqrt{3}$ | $\sqrt{3}$ | 0 | 0 |
| | 8 | −4 | 0 | 0 | 0 | 2 | 0 | 0 | $2+6\omega$ | $2+6\omega^2$ | −1 | 0 | 0 | 0 | 0 | $-\omega$ | $-\omega^2$ |
| | 8 | −4 | 0 | 0 | 0 | 2 | 0 | 0 | $2+6\omega^2$ | $2+6\omega$ | −1 | 0 | 0 | 0 | 0 | $-\omega^2$ | $-\omega$ |

Subgroup of order 324 of above group of order 1296.
Compound character $A+B+C+D$.

| Cycles | $1^9$ | $1^6 3$ | $1^5 2^2$ | $1^3 3^2$ | $1^2 2^2 3$ | $3^3$ | $3^3$ | $3^3$ | $3^3$ | 9 | 9 | 9 | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 6 | 27 | 12 | 54 | 4 | 4 | 36 | 36 | 36 | 36 | 36 | 36 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega$ | $\omega^2$ | $\omega^2$ |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega^2$ | $\omega$ | $\omega$ |
| | 6 | 3 | 2 | 0 | $-1$ | $-3$ | $-3$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | 6 | 3 | $-2$ | 0 | 1 | $-3$ | $-3$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | 12 | 0 | 0 | $-3$ | 0 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | 3 | $-1$ | 3 | $-1$ | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 4 | $-2$ | 0 | 1 | 0 | $1+3\omega$ | $1+3\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ |
| | 4 | $-2$ | 0 | 1 | 0 | $1+3\omega^2$ | $1+3\omega$ | 1 | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ |
| | 4 | $-2$ | 0 | 1 | 0 | $1+3\omega$ | $1+3\omega^2$ | $\omega$ | $\omega^2$ | $\omega^2$ | 1 | 1 | $\omega$ |
| | 4 | $-2$ | 0 | 1 | 0 | $1+3\omega^2$ | $1+3\omega$ | $\omega^2$ | $\omega$ | $\omega$ | 1 | 1 | $\omega^2$ |
| | 4 | $-2$ | 0 | 1 | 0 | $1+3\omega$ | $1+3\omega^2$ | $\omega$ | $\omega^2$ | 1 | $\omega^2$ | $\omega$ | 1 |
| | 4 | $-2$ | 0 | 1 | 0 | $1+3\omega^2$ | $1+3\omega$ | $\omega^2$ | $\omega$ | 1 | $\omega$ | $\omega^2$ | 1 |

Subgroup of order 324 of above group of order 1296.
Compound character $A+S$.

| Cycles | $1^9$ | $1^6 3$ | $1^3 6$ | $1^3 2^3$ | $1^3 2^3$ | $1^3 3^2$ | $1^3 3^2$ | $126$ | $12^4$ | $3^3$ | $3^3$ | $3^3$ | $36$ | $36$ | $36$ | $2^3 3$ | 9 |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 6 | 18 | 9 | 27 | 6 | 6 | 54 | 27 | 6 | 2 | 18 | 18 | 18 | 54 | 18 | 36 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ | $-1$ | 1 |
| | 1 | 1 | $-1$ | $-1$ | 1 | 1 | 1 | $-1$ | $-1$ | 1 | 1 | 1 | $-1$ | $-1$ | 1 | $-1$ | 1 |
| | 1 | 1 | 1 | 1 | $-1$ | 1 | 1 | $-1$ | $-1$ | 1 | 1 | 1 | 1 | 1 | $-1$ | 1 | 1 |
| | 2 | 2 | 0 | 0 | 2 | 2 | 2 | 0 | 0 | 2 | 2 | $-1$ | 0 | 0 | $-1$ | 0 | $-1$ |
| | 2 | 2 | 0 | 0 | $-2$ | 2 | 2 | 0 | 0 | 2 | 2 | $-1$ | 0 | 0 | 1 | 0 | $-1$ |
| | 2 | $-1$ | $-1$ | 2 | 0 | $-1$ | 2 | 0 | 0 | $-1$ | 2 | 2 | $-1$ | 2 | 0 | $-1$ | $-1$ |
| | 2 | $-1$ | 1 | $-2$ | 0 | $-1$ | 2 | 0 | 0 | $-1$ | 2 | 2 | 1 | $-2$ | 0 | 1 | $-1$ |
| | 6 | 3 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | $-3$ | $-3$ | 0 | $-1$ | $-1$ | 0 | $-1$ | 0 |
| | 6 | 3 | $-2$ | $-2$ | 0 | 0 | 0 | 0 | 0 | $-3$ | $-3$ | 0 | 1 | 1 | 0 | 1 | 0 |
| | 6 | 0 | 1 | $-2$ | 0 | $-3$ | 0 | 0 | 0 | 3 | $-3$ | 0 | 1 | 1 | 0 | $-2$ | 0 |
| | 6 | 0 | $-1$ | 2 | 0 | $-3$ | 0 | 0 | 0 | 3 | $-3$ | 0 | $-1$ | $-1$ | 0 | 2 | 0 |
| | 6 | $-3$ | 1 | $-2$ | 0 | 3 | 0 | 0 | 0 | 0 | $-3$ | 0 | $-2$ | 1 | 0 | 1 | 0 |
| | 6 | $-3$ | $-1$ | 2 | 0 | 3 | 0 | 0 | 0 | 0 | $-3$ | 0 | 2 | $-1$ | 0 | $-1$ | 0 |
| | 6 | 0 | 0 | 0 | 0 | 0 | $-3$ | $-1$ | 2 | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 6 | 0 | 0 | 0 | 0 | 0 | $-3$ | 1 | $-2$ | 0 | 6 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 4 | $-2$ | 0 | 0 | 0 | $-2$ | 4 | 0 | 0 | $-2$ | 4 | $-2$ | 0 | 0 | 0 | 0 | 1 |

Subgroup of order 162 of above group of order 1296.

Compound character $A+B+S+T$.

| Cycles | $1^9$ | $1^6\,3$ | $1^3\,3^2$ | $1^3\,3^2$ | $1^3\,3^2$ | $1\,2\,6$ | $1\,2\,6$ | $1\,2^4$ | $3^3$ | $3^3$ | $3^3$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 6 | 6 | 3 | 3 | 27 | 27 | 27 | 6 | 2 | 18 | 18 | 18 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | $-1$ | $-1$ | $-1$ | 1 | 1 | 1 | 1 | 1 |
| | 2 | $-1$ | $-1$ | 2 | 2 | 0 | 0 | 0 | $-1$ | 2 | $-1$ | 2 | $-1$ |
| | 2 | $-1$ | $-1$ | 2 | 2 | 0 | 0 | 0 | $-1$ | 2 | $-1$ | $-1$ | 2 |
| | 2 | $-1$ | $-1$ | 2 | 2 | 0 | 0 | 0 | $-1$ | 2 | 2 | $-1$ | $-1$ |
| | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 2 | 2 | $-1$ | $-1$ | $-1$ |
| | 3 | 0 | 0 | $3\omega$ | $3\omega^2$ | $\omega$ | $\omega^2$ | 1 | 0 | 3 | 0 | 0 | 0 |
| | 3 | 0 | 0 | $3\omega^2$ | $3\omega$ | $\omega^2$ | $\omega$ | 1 | 0 | 3 | 0 | 0 | 0 |
| | 3 | 0 | 0 | $3\omega$ | $3\omega^2$ | $-\omega$ | $-\omega^2$ | $-1$ | 0 | 3 | 0 | 0 | 0 |
| | 3 | 0 | 0 | $3\omega^2$ | $3\omega$ | $-\omega^2$ | $-\omega$ | $-1$ | 0 | 3 | 0 | 0 | 0 |
| | 6 | 3 | 0 | 0 | 0 | 0 | 0 | 0 | $-3$ | $-3$ | 0 | 0 | 0 |
| | 6 | 0 | $-3$ | 0 | 0 | 0 | 0 | 0 | 3 | $-3$ | 0 | 0 | 0 |
| | 6 | $-3$ | 3 | 0 | 0 | 0 | 0 | 0 | 0 | $-3$ | 0 | 0 | 0 |

Subgroup of order 81 of above group of order 162.

| Cycles | $1^9$ | $1^6\,3$ | $1^6\,3$ | $1^3\,3^2$ | $1^3\,3^2$ | $1^3\,3^2$ | $1^3\,3^2$ | $3^3$ | $3^3$ | $3^3$ | $3^3$ | $3^3$ | $3^3$ | $9$ | $9$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 3 | 1 | 1 | 9 | 9 | 9 | 9 | 9 | 9 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ |
| | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | 1 | 1 | $\omega^2$ | $\omega$ | 1 | 1 | $\omega^2$ | $\omega$ | 1 | 1 | $\omega^2$ | $\omega$ |
| | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | 1 | 1 | $\omega^2$ | $\omega$ | 1 | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | 1 | 1 |
| | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ | 1 | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | 1 | 1 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\omega$ | $\omega^2$ | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | $\omega^2$ | $\omega$ | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ |
| | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | 1 | 1 | $\omega$ | $\omega^2$ | 1 | 1 | 1 | 1 | $\omega$ | $\omega^2$ | $\omega^2$ | $\omega$ |
| | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | 1 | 1 | $\omega^2$ | $\omega$ | 1 | 1 | 1 | 1 | $\omega^2$ | $\omega$ | $\omega$ | $\omega^2$ |
| | 3 | 0 | 0 | 0 | 0 | $3\omega$ | $3\omega^2$ | 0 | 0 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | 0 | 0 | 0 | 0 | $3\omega^2$ | $3\omega$ | 0 | 0 | 3 | 3 | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | $1-\omega$ | $1-\omega^2$ | $\omega-\omega^2$ | $\omega^2-\omega$ | 0 | 0 | $\omega^2-1$ | $\omega-1$ | $3\omega$ | $3\omega^2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | $1-\omega^2$ | $1-\omega$ | $\omega^2-\omega$ | $\omega-\omega^2$ | 0 | 0 | $\omega-1$ | $\omega^2-1$ | $3\omega^2$ | $3\omega$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | $\omega-\omega^2$ | $\omega^2-\omega$ | $\omega^2-1$ | $\omega-1$ | 0 | 0 | $1-\omega$ | $1-\omega^2$ | $3\omega$ | $3\omega^2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | $\omega^2-\omega$ | $\omega-\omega^2$ | $\omega-1$ | $\omega^2-1$ | 0 | 0 | $1-\omega^2$ | $1-\omega$ | $3\omega^2$ | $3\omega$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | $\omega^2-1$ | $\omega-1$ | $1-\omega$ | $1-\omega^2$ | 0 | 0 | $\omega-\omega^2$ | $\omega^2-\omega$ | $3\omega$ | $3\omega^2$ | 0 | 0 | 0 | 0 | 0 | 0 |
| | 3 | $\omega-1$ | $\omega^2-1$ | $1-\omega^2$ | $1-\omega$ | 0 | 0 | $\omega^2-\omega$ | $\omega-\omega^2$ | $3\omega^2$ | $3\omega$ | 0 | 0 | 0 | 0 | 0 | 0 |

Metacyclic group of order 54, subgroup of above group of order 162.

| Cycles | $1^9$ | $1^3 3^2$ | $1^3 3^2$ | $1\,2\,6$ | $1\,2\,6$ | $1\,2^4$ | $3^3$ | $9$ | $9$ | $9$ |
|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 3 | 3 | 9 | 9 | 9 | 2 | 6 | 6 | 6 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | 1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 | 1 |
| | 1 | $\omega$ | $\omega^2$ | $\omega$ | $\omega^2$ | 1 | 1 | 1 | $\omega$ | $\omega^2$ |
| | 1 | $\omega^2$ | $\omega$ | $\omega^2$ | $\omega$ | 1 | 1 | 1 | $\omega^2$ | $\omega$ |
| | 1 | $\omega$ | $\omega^2$ | $-\omega$ | $-\omega^2$ | −1 | 1 | 1 | $\omega$ | $\omega^2$ |
| | 1 | $\omega^2$ | $\omega$ | $-\omega^2$ | $-\omega$ | −1 | 1 | 1 | $\omega^2$ | $\omega$ |
| | 2 | 2 | 2 | 0 | 0 | 0 | 2 | −1 | −1 | −1 |
| | 2 | $2\omega$ | $2\omega^2$ | 0 | 0 | 0 | 2 | −1 | $-\omega$ | $-\omega^2$ |
| | 2 | $2\omega^2$ | $2\omega$ | 0 | 0 | 0 | 2 | −1 | $-\omega^2$ | $-\omega$ |
| | 6 | 0 | 0 | 0 | 0 | 0 | −3 | 0 | 0 | 0 |

Subgroup of order 432. Compound character

$[9]+[6\,3]+[5\,2^2]+[5\,2\,1^2]+[4^2\,1]+[3^3]+[3\,2^3]+[4\,2^2\,1]+[4\,1^5]$.

| Cycles | $1^9$ | $1^3 2^3$ | $1^3 3^2$ | $1\,8$ | $1\,8$ | $1\,4^2$ | $1\,2\,6$ | $1\,2^4$ | $3^3$ | $3^3$ | $3\,6$ |
|---|---|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 36 | 24 | 54 | 54 | 54 | 72 | 9 | 8 | 48 | 72 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | 1 | −1 | −1 | 1 | 1 | 1 | 1 | 1 | −1 |
| | 2 | 0 | −1 | 0 | 0 | 2 | −1 | 2 | 2 | −1 | 0 |
| | 2 | 0 | −1 | $i\sqrt{2}$ | $-i\sqrt{2}$ | 0 | 1 | −2 | 2 | −1 | 0 |
| | 2 | 0 | −1 | $-i\sqrt{2}$ | $i\sqrt{2}$ | 0 | 1 | −2 | 2 | −1 | 0 |
| | 3 | 1 | 0 | −1 | −1 | −1 | 0 | 3 | 3 | 0 | 1 |
| | 3 | −1 | 0 | 1 | 1 | −1 | 0 | 3 | 3 | 0 | −1 |
| | 8 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | −1 |
| | 8 | −2 | 2 | 0 | 0 | 0 | 0 | 0 | −1 | −1 | 1 |
| | 16 | 0 | −2 | 0 | 0 | 0 | 0 | 0 | −2 | 1 | 0 |
| | 4 | 0 | 1 | 0 | 0 | 0 | −1 | −4 | 4 | 1 | 0 |

Subgroup of order 144 of above group of order 432.

| Cycles | $1^9$ | $1^3 2^3$ | $1\,8$ | $1\,8$ | $1\,4^2$ | $1\,4^2$ | $1\,2^4$ | $3^3$ | $3\,6$ |
|---|---|---|---|---|---|---|---|---|---|
| Order | 1 | 12 | 18 | 18 | 36 | 18 | 9 | 8 | 24 |
| | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| | 1 | −1 | −1 | −1 | 1 | 1 | 1 | 1 | −1 |
| | 1 | −1 | 1 | 1 | −1 | 1 | 1 | 1 | −1 |
| | 1 | 1 | −1 | −1 | −1 | 1 | 1 | 1 | 1 |
| | 2 | 0 | 0 | 0 | 0 | −2 | 2 | 2 | 0 |
| | 2 | 0 | $i\sqrt{2}$ | $-i\sqrt{2}$ | 0 | 0 | −2 | 2 | 0 |
| | 2 | 0 | $-i\sqrt{2}$ | $i\sqrt{2}$ | 0 | 0 | −2 | 2 | 0 |
| | 8 | 2 | 0 | 0 | 0 | 0 | 0 | −1 | −1 |
| | 8 | −2 | 0 | 0 | 0 | 0 | 0 | −1 | 1 |

## *Application to Invariant Theory*

A brief description is here given of some of the recent more significant applications of representation theory. Perhaps the most important concerns invariant theory and is a development from the ideas contained in § 10.3, page 203. For conciseness the principal results are given without proofs for which the reader is referred to the original papers in the Supplementary Bibliography.

The theory is most easily expressed in terms of *tensors*. A set of $n$ variables $x^1, x^2, \ldots, x^n$ are written with upper suffixes instead of lower. They are supposed to be subject to a group of transformations

$$x'^i = \sum \alpha^i_j x^j, \qquad x^j = \sum \beta^j_i x'^i.$$

Similar sets of variables $y^i, z^i, \ldots$, corresponding to different points in the coordinate space are transformed according to the same equations. Each set $x^i$, $y^i$, $z^i$ is said to form a contragredient tensor of rank 1.

A polynomial, say of degree 3, can then be written as

$$\sum a_{ijk} x^i x^j x^k$$

where the coefficients may be supposed to satisfy

$$a_{ijk} = a_{jik} = a_{ikj} = a_{kij} = a_{jki} = a_{kji}.$$

These coefficients transform in the following way

$$a_{ijk} = \sum \alpha^{i'}_i \alpha^{j'}_j \alpha^{k'}_k a'_{i'j'k'}.$$

This set of coefficients or any symbol with 3 lower suffixes representing a set of quantities which transform according to this equation, is called a *cogredient tensor* of *rank* 3. A cogredient tensor of rank $r$ is similarly defined, and a *contragredient* tensor, say of rank 3 transforms like

$$a^{ijk} = \sum \beta^i_{i'} \beta^j_{j'} \beta^k_{k'} a'^{i'j'k'}.$$

The tensor $a_{ijk}$ of coefficients of a cubic, by reason of the symmetry of its suffixes, is called a *symmetric tensor*. Not all tensors, however, are symmetric, e.g.

$$c_{ij} = \alpha_i \beta_j - \alpha_j \beta_i$$

satisfies $c_{ji} = -c_{ij}$, and is said to be *skew-symmetric*.

A complete tensor $b_{ij}$ of rank 2 has $n^2$ independent terms. It may be expressed as a sum of a symmetric and a skew-symmetric tensor, i.e.

$$b_{ij} = \tfrac{1}{2}(b_{ij}+b_{ji}) + \tfrac{1}{2}(b_{ij}-b_{ji}).$$

These have respectively $\frac{1}{2}n(n+1)$ and $\frac{1}{2}n(n-1)$ independent terms.

The $n^3$ terms of a tensor $b_{ijk}$ of rank 3 separate into a symmetric tensor with $\frac{1}{6}n(n+1)(n+2)$ terms, a skew-symmetric tensor with $\frac{1}{6}n(n-1)(n-2)$ terms, and other tensors which display symmetry with respect to some suffixes and skew-symmetry with respect to others.

The complete tensor of rank $r$ can be separated into subtensors by applying permutation operators to the suffixes and deriving from these the Frobenius algebra of the symmetric group on the $r$ symbols. Each irreducible idempotent of the subalgebra corresponding to the partition $(\lambda)$ of $r$, of the Frobenius algebra, when operated on the tensor $b_{i_1 \cdots i_r}$ yields a subtensor with $\chi_0^{(\lambda)}(Q_\lambda^n)/n!$ independent terms, $\chi_0^{(\lambda)}(Q_\lambda^n)/n!$ being defined as on page 126. Such a subtensor is called a tensor of type $\{\lambda\}$. It has been proved† that the matrix of transformation of these $\chi_0^{(\lambda)}(Q_\lambda^n)/n!$ terms is the invariant matrix corresponding to the partition $(\lambda)$ of the matrix of transformation $[b_i^s]$. Its spur is the $S$-function $\{\lambda\}$ of the latent roots of this matrix. The whole theory of invariant matrices may be developed in this way from the properties of tensors.

The idempotents of the Frobenius algebra are usually taken from the appropriate Young tableaux. Thus corresponding to the respective tableaux

$$(i\ j\ k), \quad \begin{pmatrix} i \\ j \\ k \end{pmatrix}, \quad \begin{pmatrix} i & j \\ k & \end{pmatrix}, \quad \left(\begin{matrix} i \\ j \end{matrix}\right.$$

the complete tensor of rank 3, $b_{ijk}$ is separated into the following subtensors for each of which the type is given on the left-hand side.

$$\{3\};\quad b_{ijk}+b_{jki}+b_{kij}+b_{kji}+b_{jik}+b_{ikj}.$$

$$\{1^3\};\quad b_{ijk}+b_{jki}+b_{kij}-b_{kji}-b_{jik}-b_{ikj}.$$

$$\{21\};\quad b_{ijk}+b_{jik}-b_{kji}-b_{kij}.$$

$$\{21\};\quad b_{ijk}+b_{kji}-b_{jik}-b_{jki}.$$

The symmetric tensor always corresponds to a partition of the form $\{n\}$, the skew-symmetric to a partition of the form $\{1^n\}$. There are $f^{(\lambda)}$ linearly independent tensors of type $\{\lambda\}$.

A tensor may have both upper and lower suffixes. A tensor with $r$ lower and $s$ upper suffixes is called a *mixed* tensor and is said to be of rank $r+s$. The important properties of tensors are as follows.

1. The product of two tensors is a tensor.

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† See Littlewood (10). References are to supplementary bibliography.

2. In a mixed tensor if one upper suffix is put equal to one lower suffix and the summation is taken for all values of the common suffix, then the result is a tensor of lower rank. Thus if $a_k^{ij}$ is a tensor, so is

$$\sum a_i^{ij} = c_j.$$

It is usual to omit the $\sum$ and assume a summation with respect to every pair of equal suffixes.

3. A tensor of rank zero is an invariant.

These properties give a ready method of constructing the various concomitants of a system of ground forms, since the tensors of the coefficients can be multiplied and contracted completely with tensors of variables so as to produce such invariant forms or concomitants. However, there is a great variety of ways of so multiplying and contracting, and sometimes different ways lead to the same concomitant, while sometimes the contracted form turns out to be identically zero. The group character method is here very useful since it gives a precise prediction of the linearly independent concomitants.

Polynomials in one set of variables correspond to symmetric tensors. For a tensor of another type the appropriate algebraic form must introduce two or more sets of variables. The first set of variables, $x^i$, can be introduced alone. The second set, $y^i$, is introduced only in the determinantal form

$$x^{ij} = \begin{vmatrix} x^i, & x^j \\ y^i, & y^j \end{vmatrix},$$

while the third set, $z^i$, is introduced in forms of the type

$$x^{ijk} = \begin{vmatrix} x^i, & x^j, & x^k \\ y^i, & y^j, & y^k \\ z^i. & z^j. & z^k \end{vmatrix},$$

and so on for determinantal forms of any order.

Thus a tensor of variables of type $\{421\}$ may be obtained from the product

$$x^{ijk}x^{pq}x^r x^s.$$

This corresponds to the Young tableau

$$\begin{pmatrix} i & p & r & s \\ j & q & & \\ k & & & \end{pmatrix}$$

The symmetry of the form corresponds to the product $NP$ as obtained from the tableau, rather than Young's form $PN$. The skew-symmetry of the columns results from the determinantal forms involved. The symmetry of the first row follows from the common value of the variable $x^i$ which appears in each determinantal form, and the set of variables $y^i$ is responsible for the symmetry of the second row.

An algebraic form of type $\{421\}$ is obtained by contracting this tensor with a tensor of coefficients, e.g.

$$a_{ijkpqrs}\, x^{ijk} x^{pq} x^r x^s.$$

The same Young tableau indicates the symmetrizing relations which hold for the coefficients $a_{ijkpqrs}$. Such an algebraic form is called a *Clebsch form.*

In $n$ variables there is a unique tensor of rank $n$ corresponding to $\{1^n\}$. It is called the alternating tensor and is denoted by

$$E_{i_1 i_2 \ldots i_n}.$$

Since it is skew-symmetric the suffixes must all be different and consist of $1, 2, 3, \ldots, n$ in some order. Corresponding to any permutation the value of the tensor is the same, save for a change of sign for a negative permutation. There is a corresponding contragredient tensor

$$E^{i_1 i_2 \ldots i_n}.$$

The following fundamental theorem has been proved.†

*Every concomitant of a system of ground forms can be obtained by multiplying and contracting tensors of the ground forms, Clebsch variable tensors, and alternating tensors.*

For concomitants linear in each of two or more ground forms the multiplication of the corresponding $S$-functions gives the types of the concomitants which appear. Further, the tableaux actually obtained in the use of the multiplication theorem for $S$-functions give the corresponding concomitants explicitly.

Thus, consider the concomitants which are linear in each of three ground forms which consist of a quartic $a_{ijkp}\, x^i x^j x^k x^p$, a cubic $b_{ijk}\, x^i x^j x^k$ and a quadratic $c_{ij}\, x^i x^j$. The concomitants correspond to the product

$$\begin{aligned}\{4\}\{3\}\{2\} &= (\{7\}+\{61\}+\{52\}+\{43\})\{2\} \\ &= \{9\}+2\{81\}+3\{72\}+\{71^2\}+3\{63\}+2\{621\}+ \\ &\quad +2\{54\}+2\{531\}+\{52^2\}+\{4^21\}+\{432\}.\end{aligned}$$

Consider the two terms $\{531\}$. One is obtained from the product

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† Littlewood (10).

$\{52\}\{2\}$, the other from the product $\{43\}\{2\}$. The tableaux obtained in the multiplications are respectively as follows:

$$\begin{pmatrix} \alpha & \alpha & \alpha & \alpha & \beta \\ \beta & \beta & \gamma & & \\ \gamma & & & & \end{pmatrix}, \quad \begin{pmatrix} \alpha & \alpha & \alpha & \alpha & \gamma \\ \beta & \beta & \beta & & \\ \gamma & & & & \end{pmatrix}.$$

The symbols in the same column correspond to suffixes which are skew-symmetric and are therefore contracted with a variable tensor such as $x^i$, $x^{ij}$, or $x^{ijk}$. Thus the two concomitants are respectively

$$a_{ijkp}\, b_{qrs}\, c_{tu}\, x^{iqt} x^{jr} x^{ku} x^{p} x^{s} \qquad \text{and} \qquad a_{ijkp}\, b_{qrs}\, c_{tu}\, x^{iqt} x^{jr} x^{ks} x^{p} x^{u}.$$

For readers familiar with the symbolic notation the interpretation is equally facile. If the ground forms are $\alpha_x^4$, $\beta_x^3$, $\gamma_x^2$, the concomitants are

$$(\alpha\beta\gamma;\, xyz)(\alpha\beta;\, xy)(\alpha\gamma;\, xy)\alpha_x \beta_x \quad \text{and} \quad (\alpha\beta\gamma;\, xyz)(\alpha\beta;\, xy)^2 \alpha_x \gamma_x.$$

For concomitants of degree 2, 3, 4 in the same ground form of type $\{\lambda\}$ the appropriate product of $S$-functions is replaced by

$$\{\lambda\}\otimes\{2\}, \quad \{\lambda\}\otimes\{3\}, \quad \{\lambda\}\otimes\{4\}.$$

The appropriate tableaux are a selection of those which occur in the products $\{\lambda\}\{\lambda\}$, $\{\lambda\}\{\lambda\}\{\lambda\}$, or $\{\lambda\}\{\lambda\}\{\lambda\}\{\lambda\}$.

There is usually little difficulty in selecting the correct tableaux. An example will illustrate. For the concomitants of a cubic the following expansions are obtained:

$$\begin{aligned} \{3\}\otimes\{2\} &= \{6\}+\{42\} \\ \{3\}\otimes\{3\} &= \{9\}+\{72\}+\{63\}+\{522\}+\{441\} \\ \{3\}\otimes\{4\} &= \{12\}+\{10.2\}+\{93\}+\{84\}+\{82^2\}+\{741\}+ \\ &\qquad +\{732\}+\{6^2\}+\{642\}+\{62^3\}+\{5421\}+\{4^3\}. \end{aligned}$$

Corresponding to the term $\{5421\}$ is obtained the following sequence of terms in $\{3\}$, $\{3\}\otimes\{2\}$, $\{3\}\otimes\{3\}$, and $\{3\}\otimes\{4\}$,

$$\{3\}, \quad \{42\}, \quad \{441\}, \quad \{5421\}.$$

The corresponding tableau is thus

$$\begin{matrix} \alpha & \alpha & \alpha & \beta & \delta \\ \beta & \beta & \gamma & \gamma & \\ \gamma & \delta & & & \\ \delta & & & & \end{matrix}$$

This tableau gives the appropriate concomitant.

The operation denoted by $\otimes$ is called the *plethysm* of $S$-functions, and the expression $\{\lambda\}\otimes\{\mu\}$ is read as '$\lambda$ plethys $\mu$'. The operation satisfies the following laws.

I. $$A\otimes(BC) = (A\otimes B)(A\otimes C).$$

In arithmetical expressions it is usually assumed that if, without any brackets, the symbols × and + appear in an expression then the operation × should precede the operation +. Similarly it is assumed that ⊗ precedes ordinary multiplication. Hence the right-hand side of the above could be written without brackets $A\otimes BA\otimes C$.

II. $$A\otimes(B+C) = A\otimes B+A\otimes C.$$

III. $$(A+B)\otimes\{\lambda\} = \sum \Gamma_{\mu\nu\lambda} A\otimes\{\mu\}B\otimes\{\nu\}$$

*where* $\Gamma_{\mu\nu\lambda}$ *is the coefficient of* $\{\lambda\}$ *in* $\{\mu\}\{\nu\}$.

IV. $$(A-B)\otimes\{\lambda\} = \sum (-1)^r\Gamma_{\mu\nu\lambda} A\otimes\{\mu\}B\otimes\{\tilde{\nu}\}$$

*where* $\{\tilde{\nu}\}$ *is the partition of* $r$ *conjugate to* $\{\nu\}$.

V. $$(AB)\otimes\{\lambda\} = \sum g_{\mu\nu\lambda} A\otimes\{\mu\}B\otimes\{\nu\}$$

*where* $\chi^{(\mu)}\chi^{(\nu)} = \sum g_{\mu\nu\lambda}\chi^{(\lambda)}$.

VI. $$(A\otimes B)\otimes C = A\otimes(B\otimes C).$$

As an example of V:

$$(\{\lambda\}\{\mu\})\otimes\{3\} = \{\lambda\}\otimes\{3\}\{\mu\}\otimes\{3\}+\{\lambda\}\otimes\{21\}\{\mu\}\otimes\{21\}+$$
$$+\{\lambda\}\otimes\{1^3\}\{\mu\}\otimes\{1^3\}.$$

Various techniques have been developed for the evaluation of $\{\lambda\}\otimes\{\mu\}$.

*Method I.* The method of generating functions is described on pages 207–8. Its main use is confined to the binary case.

*Method II.* Let $A'$ be the direct sum of a matrix $A$ and a scalar term $x$. Denote $S$-functions of $A$ and $A'$ respectively by $\{\lambda\}$ and $\{\lambda\}'$. Then

$$\{1\}' = \{1\}+x$$

and, e.g. $$\{4\}' = \{4\}+\{3\}x+\{2\}x^2+\{1\}x^3+x^4.$$

If the two sides of this equation are operated upon by $\otimes\{r\}$ various relations may be deduced which are useful in deducing the expansion of $\{4\}\otimes\{r\}$.

Firstly suppose that the expansions $\{4\}\otimes\{r\}$ are known for binary forms, and assume that $A$ is a 2-rowed matrix. The expansion of

$\{4\}'\otimes\{3\}$ is found as follows. The term independent of $x$ on the right is

$$\{4\}\otimes\{3\} = \{12\}+\{10.2\}+\{93\}+\{84\}+\{6^2\}.$$

Hence $\{4\}'\otimes\{3\}$ contains the terms

$$\{12\}'+\{10.2\}'+\{93\}'+\{84\}'+\{6^2\}'.$$

The coefficient of $x$ is

$$\{4\}\otimes\{2\}\{3\} = \{11\}+\{10.1\}+2\{92\}+2\{83\}+2\{74\}+\{65\}.$$

All these terms except for $\{74\}$ are accounted for by the expansions of the known terms in $\{4\}'\otimes\{3\}$. To account for the extra term $\{74\}$, $\{4\}'\otimes\{3\}$ must also contain the term $\{741\}'$.

Similarly, expanding the coefficient of $x^2$ which is

$$\{4\}\otimes\{2\}\{2\}+\{4\}\{3\}\otimes\{2\},$$

the terms unaccounted for are

$$\{82\}+\{64\}.$$

Hence $\{4\}'\otimes\{3\}$ includes

$$\{822\}'+\{642\}'.$$

Continuation of this procedure gives

$$\{4\}'\otimes\{3\} = \{12\}'+\{10.2\}'+\{93\}'+\{84\}'+\{6^2\}'+\{741\}'+\{82^2\}'+ \\ +\{642\}'+\{4^3\}'.$$

When the expansions are known for ternary forms the same procedure gives the results for quaternary forms.

Alternatively, by commencing with the highest power of $x$ the same procedure gives the expansions for $n$-ary forms directly.

*Method III.* Let it be supposed that the number of variables is always large enough so that no $S$-functions are identically zero. Consider the matrices $A$, $A'$ as in Method II, with

$$\{1\}' = \{1\}+x.$$

Then generally $$\{\lambda\}' = \{\lambda\}+\sum \Gamma_{r\mu\lambda}\{\mu\}x^r,$$

where $\Gamma_{r\mu\lambda}$ is the coefficient of $\{\lambda\}$ in $\{r\}\{\mu\}$. Suppose that $\{\lambda\}\otimes\{n-1\}$ is known. Then the coefficient of $x$ in $\{\lambda\}'\otimes\{n\}$ is

$$\{\lambda\}\otimes\{n-1\}(\sum \Gamma_{1\mu\lambda}\{\mu\}).$$

If $$\{\lambda\}'\otimes\{n\} = \sum \{\nu\}',$$

then the coefficient of $x$ in this, which is equal to

$$\{\lambda\}\otimes\{n-1\}(\sum \Gamma_{1\mu\lambda}\{\mu\})$$

is also equal to $\sum \Gamma_{1\zeta\nu}\{\zeta\}$.

For small degrees if $\sum \Gamma_{1\zeta\nu}\{\zeta\}$ is known then $\sum \{\nu\}$ may easily be inferred, and this is the required expansion of $\{\lambda\}\otimes\{n\}$.

As an example the expansion of $\{3\}\otimes\{4\}$ will be found, it being given that

$$\{3\}\otimes\{3\} = \{9\}+\{72\}+\{63\}+\{52^2\}+\{4^21\}.$$

In $\{3\}\otimes\{3\}\{2\}$, $S$-functions occur corresponding to the following partitions:

11, 10.1, 92, 92, 83, 83, 821, 74, 74, 731, 731, 722, 722, 65, 641, 641, 632, 632, 6221, 542, 542, 5411, 5321, 5222, 443, 4421.

These are associated with $S$-functions of weight 12 in the correspondence $\{\nu\} \rightarrow \sum \Gamma_{1\zeta\nu}\{\zeta\}$ in the following way:

$$\{12\}\rightarrow\{11\},\ \{10.2\}\rightarrow\{10.1\}+\{92\},\ \{93\}\rightarrow\{92\}+\{83\},$$
$$\{84\}\rightarrow\{83\}+\{74\},$$
$$\{822\}\rightarrow\{821\}+\{722\},\ \{741\}\rightarrow\{74\}+\{731\}+\{641\},$$
$$\{732\}\rightarrow\{731\}+\{722\}+\{632\},\ \{66\}\rightarrow\{65\},$$
$$\{642\}\rightarrow\{641\}+\{632\}+\{542\},$$
$$\{6222\}\rightarrow\{6221\}+\{5222\},$$
$$\{5421\}\rightarrow\{542\}+\{5411\}+\{5321\}+\{4421\},\ \{444\}\rightarrow\{443\}.$$

Thus

$$\{3\}\otimes\{4\} = \{12\}+\{10.2\}+\{93\}+\{84\}+\{82^2\}+\{741\}+\{732\}+$$
$$+\{6^2\}+\{642\}+\{62^3\}+\{5421\}+\{4^3\}.$$

As the degrees get larger certain alternatives present themselves and it becomes difficult to select the correct $\{\nu\}$. However, the method may still be valuable if used in conjunction with some other method which would indicate the correct choice when difficulty arises.

### *Concomitant Types*[†]

Consider the concomitants which are linear in each of three ground forms, each of the same type $\{\lambda\}$. The concomitants correspond to the terms in the expansion of

$$\{\lambda\}\{\lambda\}\{\lambda\} = \{\lambda\}\otimes\{1\}^3 = \{\lambda\}\otimes\{3\}+2\{\lambda\}\otimes\{21\}+\{\lambda\}\otimes\{1^3\}.$$

The concomitants corresponding to the terms in $\{\lambda\}\otimes\{3\}$ are symmetric in the ground forms. If the ground forms were made identical they would become the concomitants of degree 3 in a single ground form. In fact these concomitants linear in the three ground forms can be obtained from the concomitants of degree 3 in the first ground form by *polarization* with respect to the second and third ground forms. Separate concomitants which can be obtained from one another by polarization with respect to different ground forms are classed together to give a single *concomitant type*. The terms in $\{\lambda\}\otimes\{3\}$ correspond to those concomitant types which are symmetric in the ground forms. They are said to be of *class* $\{3\}$.

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[†] Littlewood (12). See also Todd (22).

The terms in $\{\lambda\}\otimes\{1^3\}$ are alternating in the three ground forms. They would become zero if two ground forms were made equal. These give the *alternating concomitant types* or the concomitant types of class $\{1^3\}$.

Similarly the terms in $\{\lambda\}\otimes\{21\}$ give the concomitant types of class $\{21\}$. Such terms appear twice in the full list of concomitants linear in the three ground forms. But for each term the two concomitants belong to the same concomitant type and can be obtained from one another by permuting the ground forms. If the ground forms are $f$, $g$, $h$, then if standard Young tableaux are formed with these symbols

$$\begin{pmatrix} f & g \\ h & \end{pmatrix}, \qquad \begin{pmatrix} f & h \\ g & \end{pmatrix},$$

to the left tableau correspond concomitants which are symmetric in $f$ and $g$, and skew-symmetric in $f$ and $h$, while for the tableau on the right the roles of $g$ and $h$ are reversed.

More generally the complete set of concomitants linear in $n$ ground forms each of type $\{\lambda\}$ may be analysed into concomitant types, the concomitant types of class $\{\mu\}$ where $(\mu)$ is a partition of $n$ corresponding to the terms in $\{\lambda\}\otimes\{\mu\}$.

The invariants of systems of quadrics have been analysed in this way.†

*Invariant-theory under Restricted Groups*‡

The full linear group of transformations on $n$ variables possesses various sub-groups, each leaving a given system of forms invariant. Such a system of forms must be infinite since all powers of an invariant form are invariant and indeed every concomitant of a set of invariant forms. However, a finite set called the *fundamental forms* may be found such that every invariant form is a concomitant of these. A fundamental theorem for such restricted groups is as follows.‡

*Every concomitant of a set of ground forms under a restricted group* *may be obtained by multiplying and contracting ground form tensors, fundamental form tensors, variable tensors, and alternating tensors.*

The most important restricted group is the orthogonal group, for which the one fundamental form is a non-singular quadratic called the *metric*. For the rotation group the alternating tensor also is invariant.

The method of characteristic analysis can be applied to the orthogonal group by the use of plethysm with respect to the characters of the group. In the calculation use is made of $S$-functions and the laws of plethysm. Thus the concomitants of a ternary quadratic under the orthogonal group may be found as follows.

$$\begin{aligned}[2]\otimes\{2\} &= (\{2\}-\{0\})\otimes\{2\} = \{2\}\otimes\{2\}-\{2\} \\ &= \{4\}+\{2^2\}-\{2\} = \{4\} \\ &= [4]+[2]+[0].\end{aligned}$$

The term [4] corresponds to the square of the ground form and is reducible. The quadratic and invariant are irreducible.

$$\begin{aligned}[2]\otimes\{3\} &= (\{2\}-\{0\})\otimes\{3\} = \{2\}\otimes\{3\}-\{2\}\otimes\{2\} \\ &= \{6\}+\{42\}+\{2^3\}-\{4\}-\{2^2\}.\end{aligned}$$

It will be shown that

$$\{42\} = [4]+[3]+2[2]+[0].$$

Hence $$[2]\otimes\{3\} = [6]+[4]+[3]+[2]+[0].$$

Reducible concomitants correspond to [6]+[4]+[2]. The irreducibles are a cubic and an invariant.

It is necessary sometimes in this analysis, when there are $n = 2\nu$ or $n = 2\nu+1$ variables, to express an $S$-function corresponding to a partition with $> \nu$ parts in terms of orthogonal group characters. Theorem II, page 240, is only available when there are $\leqslant \nu$ parts. The following method may be used for $> \nu$ parts.

If $n = 2\nu$ express the $S$-function in the form

$$\{r+\lambda_1, r+\lambda_2,\ldots, r+\lambda_\nu, r-\mu_\nu,\ldots, \nu-\mu_1\},$$

or if $n = 2\nu+1$ in the form

$$\{r+\lambda_1, r+\lambda_2,\ldots, r+\lambda_\nu, r, r-\mu_\nu,\ldots, r-\mu_1\}.$$

Ignoring a change of sign for a transformation of negative determinants the value of such an $S$-function is independent of $r$ and is denoted by $\{\lambda; \mu\}$. The following formula† allows $\{\lambda; \mu\}$ to be expressed in terms of smaller partitions, so that repetition of the procedure allows the required expansion to be obtained. If $(\delta)$ denotes any partition of any number $d$, and $(\epsilon)$ is the conjugate partition, the formula is

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† Littlewood (11).

$$\{\lambda;\,\mu\} = \{\lambda\}\{\mu\}+\sum(-1)^d\Gamma_{\alpha\delta\lambda}\Gamma_{\beta\epsilon\mu}\{\alpha\}\{\beta\}.$$

The summation is with respect to all partitions $\delta$ of all numbers $d$ such that $\{\lambda\}$ appears with coefficient $\Gamma_{\alpha\delta\lambda}$ in $\{\alpha\}\{\delta\}$ and $\{\mu\}$ with coefficient $\Gamma_{\beta\epsilon\mu}$ in $\{\beta\}\{\epsilon\}$.

Thus for ternary orthogonal forms

$$\begin{aligned}\{42\} &= \{2;\,2\} = \{2\}\{2\}-\{1\}\{1\}\\ &= \{4\}+\{31\}+\{2^2\}-\{2\}-\{1^2\}.\\ \{31\} &= \{2;\,1\} = \{2\}\{1\}-\{1\}\\ &= \{3\}+\{21\}-\{1^2\}.\\ \{21\} &= \{1;\,1\} = \{1\}\{1\}-\{0\} = [2]+[1].\end{aligned}$$

Thus $\{31\} = [3]+[2]+[1]$,

and $\{42\} = [4]+[3]+2[2]+[0]$.

*The Symplectic Group*†

The group which leaves invariant a non-singular linear complex, of type $\{1^2\}$, is called the symplectic group. For the linear complex to be non-singular $n$ must be even, say $n = 2\nu$. The invariant form is usually expressed in canonical form as

$$(x_1y_2-x_2y_1)+(x_3y_4-x_4y_3)+\ldots+(x_{n-1}y_n-x_ny_{n-1}).$$

The group has properties very similar to those of the orthogonal group, but it is simpler, for the alternating tensor is a concomitant of the fundamental tensor and hence the determinant is always $+1$. No discrimination need be made then, as between orthogonal group and rotation group. Also there is nothing corresponding to spinors.

As with the orthogonal group there is a character corresponding to every partition $(\lambda)$ into $\leqslant \nu$ parts. This character is denoted by $\langle\lambda\rangle$. $S$-functions with $\leqslant \nu$ parts may be expressed in terms of these characters and conversely by the formulae

$$\begin{aligned}\{\lambda\} &= \langle\lambda\rangle+\sum\Gamma_{\mu\beta\lambda}\langle\mu\rangle\\ \langle\lambda\rangle &= \{\lambda\}+\sum(-1)^p\Gamma_{\mu\alpha\lambda}\{\mu\}.\end{aligned}$$

The partitions $(\alpha)$ and $(\beta)$ are summed for all the partitions indicated on page 238, (11·9; 1) and (11·9; 2), $\alpha$ being a partition of $2p$.

† Ibid.

Concomitants under the symplectic group are then obtained in the same way as for the orthogonal group. Thus for a quadratic

$$\langle 2\rangle\otimes\{2\} = \{4\}+\{2^2\} = \langle 4\rangle+\langle 2^2\rangle+\langle 1^2\rangle+\langle 0\rangle.$$

The irreducible concomitants correspond to $\langle 2^2\rangle$, $\langle 1^2\rangle$, $\langle 0\rangle$.

*Spinors*†

Consider once again the orthogonal group. The spinors are not associated with algebraic forms in the same way as are the tensors. The theory of concomitants, however, need not refer to algebraic forms at all. The concomitants could be tensors which are polynomials in the components of the ground form tensors. The theory is unaltered. But from this point of view it is possible to regard a spinor as a ground form.

The concomitants of a basic spinor may be obtained from first principles from the latent roots of the transforming matrix.

*3 Variables.* If $[1] = 1+2\cos\theta$, then $[\frac{1}{2}] = e^{\frac{1}{2}i\theta}+e^{-\frac{1}{2}i\theta}$. Hence

$$\begin{aligned}[\tfrac{1}{2}]\otimes\{n\} &= e^{\frac{1}{2}ni\theta}+e^{\frac{1}{2}(n-2)i\theta}+\ldots+e^{-\frac{1}{2}ni\theta}\\ &= [\tfrac{1}{2}n].\end{aligned}$$

The square of a basic spinor may be expressed as a vector. It is convenient to regard this concomitant of degree 2 and type [1] as reducible. The ternary basic spinor, then, has no irreducible concomitant apart from the ground form itself.

The above result may be used as follows to find the concomitants of any ternary orthogonal form if the concomitants of binary forms are known. Thus for a ternary orthogonal $p$-ic

$$[p]\otimes\{n\} = [\tfrac{1}{2}]\otimes\{2p\}\otimes\{n\},$$

and this may be calculated if $\{2p\}\otimes\{n\}$ is known. The concomitants of a ternary orthogonal cubic, for example, may be deduced from those of a binary sextic. Since for binary forms

$$\begin{aligned}\{6\}\otimes\{3\} &= \{18\}+\{16.2\}+\{15.3\}+\{14.4\}+\{13.5\}+2\{12.6\}+\{10.8\}\\ &= \{18\}+\{14\}+\{12\}+\{10\}+\{8\}+2\{6\}+\{2\},\end{aligned}$$

therefore for ternary orthogonal forms

$$[3]\otimes\{3\} = [9]+[7]+[6]+[5]+[4]+2[3]+[1].$$

The correspondence is complete and irreducible concomitants in the binary case correspond to irreducible concomitants in the ternary orthogonal case.

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† Littlewood (13), (15).

There is, in fact, a complete isomorphism between the binary group with unit determinant and the ternary orthogonal group. It is not simple but is a 2 : 1 isomorphism because of the 2-valued nature of spinors.

*4 variables.* It is convenient to deal with the rotation group. The full orthogonal character $[p, q]$ with $q > 0$ separates into two rotation group characters which are conveniently denoted by $[p, q]$ and $[p, -q]$. The two parts of a basic spinor are $[\frac{1}{2}, \frac{1}{2}]$ and $[\frac{1}{2}, -\frac{1}{2}]$.

If
$$[1] = 2\cos\theta + 2\cos\phi$$
then
$$[\tfrac{1}{2}, \tfrac{1}{2}] = 2\cos\tfrac{1}{2}(\theta+\phi), \qquad [\tfrac{1}{2}, -\tfrac{1}{2}] = 2\cos\tfrac{1}{2}(\theta-\phi).$$
It may be deduced that
$$[\tfrac{1}{2}, \tfrac{1}{2}] \otimes \{n\} = [\tfrac{1}{2}n, \tfrac{1}{2}n], \qquad [\tfrac{1}{2}, -\tfrac{1}{2}] \otimes \{n\} = [\tfrac{1}{2}n, -\tfrac{1}{2}n],$$
and more generally
$$[\tfrac{1}{2}, \tfrac{1}{2}] \otimes \{p\}[\tfrac{1}{2}, -\tfrac{1}{2}] \otimes \{q\} = [\tfrac{1}{2}(p+q), \tfrac{1}{2}(p-q)].$$

The concomitants under the quaternary rotation group can then be shown to correspond to concomitants of a corresponding form under a *double binary group*.

Let $\{\lambda\}$ denote an $S$-function of the latent roots of the 2-rowed basic spin matrix of type $[\frac{1}{2}, \frac{1}{2}]$, and $\{\mu\}'$ similarly for the type $[\frac{1}{2}, -\frac{1}{2}]$. Then the characters of the double binary group correspond to the products $\{\lambda\}\{\mu\}'$.

Thus, since
$$\begin{aligned}\{3\}\{1\}' \otimes \{2\} &= \{3\} \otimes \{2\}\{1\}' \otimes \{2\} + \{3\} \otimes \{1^2\}\{1\}' \otimes \{1^2\} \\ &= (\{6\}+\{42\})\{2\}' + (\{51\}+\{33\})\{1^2\}' \\ &= \{6\}\{2\}' + \{2\}\{2\}' + \{4\} + \{0\},\end{aligned}$$
thence for the quaternary orthogonal group
$$\begin{aligned}[21] \otimes \{2\} &= ([\tfrac{1}{2}, \tfrac{1}{2}] \otimes \{3\}[\tfrac{1}{2}, -\tfrac{1}{2}] \otimes \{1\}) \otimes \{2\} \\ &= [\tfrac{1}{2}, \tfrac{1}{2}] \otimes \{6\}[\tfrac{1}{2}, -\tfrac{1}{2}] \otimes \{2\} + [\tfrac{1}{2}, \tfrac{1}{2}] \otimes \{2\}[\tfrac{1}{2}, -\tfrac{1}{2}] \otimes \{2\} + \\ &\qquad + [\tfrac{1}{2}, \tfrac{1}{2}] \otimes \{4\} + [0] \\ &= [42] + [2] + [2^2] + [0].\end{aligned}$$

There is a 2 : 1 isomorphism between the quaternary rotation group and the double binary group. The correspondence is complete as between reducible or irreducible concomitants, provided that reducibility is suitably defined. A concomitant is reducible if it is the principal part of the product of two concomitants, or is a linear combination of such principal parts. It is convenient then, to define the principal part of the product of two forms of types $[\lambda_1, \lambda_2]$ and $[\mu_1, \mu_2]$ as a form of type $[\lambda_1+\mu_1, \lambda_2+\mu_2]$. This definition is quite conventional if $\lambda_2$ and $\mu_2$ are of the same sign, but may appear unusual if they are of opposite sign. Thus the principal part of the product of the two irreducible parts of a 6-vector, of types $[1, 1]$ and $[1, -1]$ respectively is just a quadratic of type $[2]$. Such a definition, however, is the only one which gives consistent results.

For the full orthogonal group, results may be calculated in terms of the rotation group characters. Thus using $[\lambda]'$ to denote a character of the full orthogonal group, then for a form of type $[21]'$

$$\begin{aligned}[21]'\otimes\{2\} &= ([21]+[2,-1])\otimes\{2\} \\ &= [21]\otimes\{2\}+[21][2,-1]+[2,-1]\otimes\{2\}.\end{aligned}$$

The product $[21][2,-1]$ corresponds to

$$\begin{aligned}\{3\}\{1\}'\{1\}\{3\}' &= (\{4\}+\{31\})(\{4\}'+\{31\}') \\ &= \{4\}\{4\}'+\{4\}\{31\}'+\{31\}\{4\}'+\{31\}\{31\}'.\end{aligned}$$

Thus $$[21][2,-1] = [4]+[31]+[3,-1]+[2].$$

Hence

$$\begin{aligned}[21]'\otimes\{2\} &= [42]+[2^2]+[2]+[0]+[4]+[31]+[3,-1]+[2]+ \\ &\qquad +[4,-2]+[2,-2]+[2]+[0] \\ &= [42]'+[2^2]'+[31]'+[4]'+3[2]'+2[0]'.\end{aligned}$$

*5 variables*

In 5 variables $$[\tfrac{1}{2},\tfrac{1}{2}]\otimes\{2\} = [1^2],$$ $$[\tfrac{1}{2},\tfrac{1}{2}]\otimes\{1^2\} = [1]+[0].$$

The last equation implies that a form which is a linear complex in the 4 components of a basic spinor, i.e. a form of type $\{1^2\}$ in these components, possesses an invariant. There exists an invariant form of type $\{1^2\}$ in the components of basic spinors. This implies that the group of transformation of these 4 components is either the symplectic group or a subgroup thereof. It is therefore valid to use the symbol $\otimes$ followed by a symplectic group character. Thus

$$[\tfrac{1}{2},\tfrac{1}{2}]\otimes\langle 2\rangle = [1^2],\ [\tfrac{1}{2},\tfrac{1}{2}]\otimes\langle 1^2\rangle = [1].$$

More generally

$$[\tfrac{1}{2},\tfrac{1}{2}]\otimes\langle m,n\rangle = [\tfrac{1}{2}(m+n),\tfrac{1}{2}(m-n)].$$

There is a 2 : 1 correspondence between the 5-variable orthogonal group and the 4-variable symplectic group. The forms under the two groups are in complete correspondence, a form of type $\langle m,n\rangle$ corresponding to a form of type $[\frac{1}{2}(m+n),\frac{1}{2}(m-n)]$. Any knowledge of concomitants under the one group implies a similar knowledge of concomitants of the corresponding forms under the other group.

*6 variables*

The 6-variable rotation group proves in a similar way to be isomorphic with the 4-variable full linear group. Thus

$$[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]\otimes\{2\} = [111], \qquad [\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]\otimes\{1^2\} = [1],$$
$$[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]\otimes\{1^3\} = [\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}], \qquad [\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]\otimes\{1^4\} = [0].$$

More generally

$$[\tfrac{1}{2},\tfrac{1}{2},\tfrac{1}{2}]\otimes\{pqr\} = [\tfrac{1}{2}(p+q-r),\tfrac{1}{2}(p-q+r),\tfrac{1}{2}(p-q-r)].$$

*8 variables*

A similar procedure with the 8-variable rotation group leads to an unusual automorphism of this group for which spinors and true representations are interchanged. Denote $[\frac{1}{2},\frac{1}{2},\frac{1}{2},\frac{1}{2}]$ and $[\frac{1}{2},\frac{1}{2},\frac{1}{2},-\frac{1}{2}]$ by $\Delta$ and $\Delta'$ respectively. Firstly

$$\Delta\otimes\{2\} = [1^4]+[0].$$

The 8 components of one conjugate part of a basic spinor therefore possess a quadratic invariant. It is valid therefore to follow $\otimes$ by a character of the orthogonal group. Thence

$$\Delta\otimes[2] = [1^4],$$
$$\Delta\otimes[1^2] = [1^2],$$
$$\Delta\otimes[1^3] = [\tfrac{3}{2},\tfrac{1}{2},\tfrac{1}{2},-\tfrac{1}{2}],$$
$$\Delta\otimes\{1^4\} = [2]+[111,-1].$$

For the 8-variable rotation group $\{1^4\}$ separates into two conjugate parts as

$$\{1^4\} = [1111]+[111,-1].$$

The choice as to which of the two symbols shall correspond to which of the two conjugate parts is arbitrary. It may therefore be assumed that

$$\Delta\otimes[1111] = [111,-1],$$
$$\Delta\otimes[111,-1] = [2].$$

Again, since the spin space is orthogonal, it will have its own spinors and the symbol $\otimes$ may be followed by a basic spinor. This gives

$$\Delta \otimes \Delta = \Delta',$$

and $$\Delta' \otimes \Delta = \Delta \otimes \Delta' = \Delta \otimes \Delta \otimes \Delta = [1].$$

For the most general partition (into integers or into 4 halves of odd integers)

$$\Delta \otimes [pqrs]$$
$$= [\tfrac{1}{2}(p+q+r-s),\ \tfrac{1}{2}(p+q-r+s),\ \tfrac{1}{2}(p-q+r+s),\ \tfrac{1}{2}(p-q-r-s)].$$

The automorphism corresponding to $\Delta \otimes$ is of index 3. There is also an automorphism which interchanges $[p, q, r, s]$ with $[p, q, r, -s]$. This is, of course, of index 2. The 8-variable rotation group has, therefore, a group of outer automorphisms simply isomorphic with the symmetric group on 3-symbols.

The automorphisms may be employed to obtain the concomitants of a spinor from those of a true tensor. Thus if concomitants of a form of type $[1^3]$ are known, the concomitants of a spinor of type $[\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}]$ may be deduced, corresponding to the relation

$$[\tfrac{3}{2}, \tfrac{1}{2}, \tfrac{1}{2}, \tfrac{1}{2}] \otimes \{n\} = \Delta' \otimes [1^3] \otimes \{n\}.$$

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(1) 43. Frobenius (30)–(40).
(2) 53. Schur (64).
(3) 56. Frobenius (36).
(4) 58. Frobenius (31).

### CHAPTER V

(1) 59. See MacMahon (6), vol. i, Chapter I. For the 'Ferrers-Sylvester' graph, ibid., p. 124.
(2) 60. Frobenius (34).
(3) 61. See MacMahon (6), vol. i, p. 45.
(4) 61. Frobenius (34).
(5) 67. Littlewood and Richardson (49).
(6) 67. See MacMahon (6), p. 124.

### CHAPTER VI

(1) 81. Littlewood and Richardson (49). The name 'immanant' was suggested by Prof. A. R. Forsyth.
(2) 84. So named by Littlewood and Richardson (49).

(3) 87. Jacobi, see Muir (8), vol. i, p. 341.
Trudi, see Muir (8), vol. iii, p. 135.
Naegelsbach, see Muir (8), vol. iii, p. 144.
Kostka, see Muir (8), vol. iv, pp. 145, 154, 158.
(4) 87. Schur (63).
(5) 89. Littlewood and Richardson (49), p. 109.
(6) 91. The essential result, Theorem V, was stated without proof for the general case by Littlewood and Richardson (49). It was proved by Robinson (62). See also Murnaghan (55), (56).
(7) 98. Littlewood (45).
(8) 98. Murnaghan (53).
(9) 102. e.g. see E. B. van Vleck, 'Selected topics in the theory of divergent series and continued fractions', *The Boston Colloquium* (1903). See also J. Hadamard, *J. de Math.* (4), 8 (1892), 118, 138, who uses both this special form and also the determinant for the more general $S$-function.
(10) 118. Littlewood and Richardson (49).
(11) 120. The relation between the two equations was pointed out by MacMahon (52).

## CHAPTER VII

(1) 122. Littlewood and Richardson (51).
(2) 122. Littlewood and Richardson (50).

## CHAPTER VIII

(1) 137. Frobenius (34).
(2) 138. Littlewood and Richardson (51).
(3) 140. Littlewood and Richardson (49).
(4) 141. Murnaghan (54).
(5) 142. Littlewood and Richardson (50).
(6) 143. Littlewood and Richardson (50).

## CHAPTER IX

(1) 147. Littlewood (44). Many of the properties obtained here are to be found in Frobenius's papers (30)–(38).
(2) 164. Redfield (60).
(3) 165. This is a simplification, elaboration, and extension of a method given by Redfield (60).

## CHAPTER X

(1) 178. Schur (63).
(2) 182. Schur (63).
(3) 183. Littlewood (48).
(4) 193. Schur (63).
(5) 193. Aitken (17); Littlewood (46).
(6) 195. Aitken (17).
(7) 203. Littlewood (47), see also Weyl (14).
(8) 208. Young (75).

## CHAPTER XI

(**1**) 211. Schur (66). Hurwitz (41) originally applied the method of group integration to invariant theory.
(**2**) 214. Brauer and Schur (21).
(**3**) 215. Auerbach (18).
(**4**) 215. Schur (66).
(**5**) 217. Weyl (73).
(**6**) 223. Schur (66).
(**7**) 238. Hitherto unpublished.
(**8**) 243. Brauer (19).
(**9**) 246. See Murnaghan (9), p. 288.
(**10**) 247. Hitherto unpublished.
(**11**) 248. Brauer and Weyl (22).
(**12**) 251. See Eddington (26), (27), Newman (58), Littlewood (43).
(**13**) 256. Hitherto unpublished.

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